

DIRAC LIE GROUPS

D. LI-BLAND AND E. MEINRENKEN

ABSTRACT. A classical theorem of Drinfel'd states that the category of simply connected Poisson Lie groups H is isomorphic to the category of Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, where \mathfrak{h} is the Lie algebra of H . In this paper, we consider *Dirac Lie groups*, that is, Lie groups H endowed with a multiplicative Courant algebroid \mathbb{A} and a Dirac structure $E \subseteq \mathbb{A}$ for which the multiplication is a Dirac morphism. It turns out that the simply connected Dirac Lie groups are classified by so-called *Dirac Manin triples*. We give an explicit construction of the Dirac Lie group structure defined by a Dirac Manin triple, and develop its basic properties.

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0. INTRODUCTION

Dirac structures were introduced by T. Courant [6] as a common framework for closed 2-forms and Poisson structures on manifolds. He showed that the integrability condition $d\omega = 0$ for 2-forms and $[\pi, \pi] = 0$ for bivector fields admits a common generalization to an integrability condition on Lagrangian subbundles $E \subseteq TM = TM \oplus T^*M$ relative to a certain bracket on $\Gamma(TM)$. Liu-Weinstein-Xu [21] generalized Courant's original set-up, replacing TM with a more general notion of a *Courant algebroid* $\mathbb{A} \rightarrow M$. The theory of Courant algebroids and Dirac structures was clarified and simplified in the work of Dorfman [7], Ševera [36, Letter no.7], Roytenberg [34], Uchino [38], and others. It has recently gained attention through the development of generalized complex geometry [11, 13], and it provides a unified setting for various types moment maps [1, 5].

A Poisson Lie group is a Lie group H , equipped with a Poisson structure such that the multiplication map is Poisson. To extend this definition to Dirac geometry, it is required that the Courant algebroid \mathbb{A} itself carries a multiplicative structure. As suggested by Mehta [27] and further explored in [20], we require that \mathbb{A} carries a \mathcal{VB} -groupoid structure $\mathbb{A} \rightrightarrows \mathfrak{g}$ over the group $H \rightrightarrows \text{pt}$, in such a way that the groupoid multiplication is a Courant morphism $\text{Mult}_{\mathbb{A}}: \mathbb{A} \times \mathbb{A} \dashrightarrow \mathbb{A}$. (For the standard Courant algebroid $\mathbb{A} = \mathbb{T}H$ this structure is automatic, with $\mathfrak{g} = \mathfrak{h}^*$.) One then has a notion of a *multiplicative* Dirac structure $E \subseteq \mathbb{A}$. In the case of $\mathbb{T}H$ these were classified in the work of Ortiz [30] and Jotz [15], independently. While [15, 30] refer to multiplicative Dirac structures as Dirac Lie group structures, we will reserve this latter term for the case that the multiplication map is a Dirac morphism (i.e a morphism of Manin pairs as in [5]). For $\mathbb{A} = \mathbb{T}H$, only the Poisson Lie group structures are Dirac Lie group structures in our sense, but many more examples are obtained by considering more general Courant algebroids. These include the well known Cartan-Dirac structure (cf. [1] and references therein), and the examples in Section 5 of [16]. One of the goals of this paper is to develop the theory of Dirac Lie groups in this setting. The super-geometric interpretation of Dirac Lie group structures was previously studied in [20] under the name *MP Lie groups*.

By Drinfel'd's theorem [8], the category of simply connected Poisson Lie groups H is canonically equivalent to the category of Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. That is, \mathfrak{d} is a Lie algebra with a vector space decomposition into two Lie subalgebras $\mathfrak{g}, \mathfrak{h}$, and equipped with a non-degenerate invariant symmetric bilinear form ('metric') for which $\mathfrak{g}, \mathfrak{h}$ are Lagrangian.

According to a recent result of Michal Siran [37], the non-simply connected Poisson Lie groups are similarly classified by H -equivariant Manin triples.

We will show that Dirac Lie groups H are classified by H -equivariant *Dirac Manin triples* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. These consist of a Lie algebra \mathfrak{d} with a vector space direct sum decomposition into two Lie subalgebras $\mathfrak{g}, \mathfrak{h}$, together with an invariant symmetric bilinear form β on the dual space \mathfrak{d}^* such that \mathfrak{g} is β -coisotropic, i.e. the restriction of β to $\text{ann}(\mathfrak{g})$ is zero. Here β may be degenerate or even zero, and there is no compatibility requirement between β and \mathfrak{h} . We will prove:

Theorem 0.1. *The category of Dirac Lie groups and the category of equivariant Dirac Manin triples are canonically equivalent.*

Theorem 0.1 may be deduced from the classification of *MP Lie groups* in [20], but we will give a direct proof, not using super geometry.

The functor from Dirac Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ to Dirac Lie groups is constructed as follows. As a first step, we use a reduction procedure to construct a new Dirac Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$, with a Lie algebra homomorphism $f: \mathfrak{q} \rightarrow \mathfrak{d}$ taking \mathfrak{r} to \mathfrak{h} and restricting to the identity on \mathfrak{g} . The new Dirac Manin triple is such that γ is non-degenerate and \mathfrak{g} is Lagrangian in \mathfrak{q} . The corresponding Dirac Lie group (\mathbb{A}, E) is described using a ‘left trivialization’

$$\mathbb{A} = H \times \mathfrak{q}, \quad E = H \times \mathfrak{g},$$

where $H \times \mathfrak{q}$ is an *action Courant algebroid* [19]. An explicit description of the groupoid structure in terms of this trivialization is given in Theorem 5.2. It is rather cumbersome, however, to verify the compatibility properties from these explicit formulas. Therefore, we show that one can also obtain (\mathbb{A}, E) by co-isotropic reduction of the direct product of the multiplicative Manin pairs (TH, TH) and $(\bar{\mathfrak{q}} \oplus \mathfrak{q}, \mathfrak{g} \oplus \mathfrak{g})$ (where $\bar{\mathfrak{q}} \oplus \mathfrak{q} \rightrightarrows \mathfrak{q}$ carries the pair groupoid structure).

Of particular interest are the Dirac Lie group structures (\mathbb{A}, E) over H for which the underlying Courant algebroid is *exact*, in the sense of Ševera. We prove that this is the case if and only if β is non-degenerate and \mathfrak{g} is Lagrangian. In this case we construct a *canonical* isomorphism with the Courant algebroid TH_η , with twisting 3-form the Cartan 3-form over H . We hence obtain another concrete description of the Dirac Lie group structure, in terms of differential forms and vector fields.

The organization of the paper is as follows. In Section 1 we summarize the basic theory of Courant algebroids, Dirac structures, and their morphisms. In Section 2 we introduce and motivate our definition of Dirac Lie groups. Next, in Sections 3 and 4 we classify Dirac Lie groups and their morphisms in terms of Lie theoretic data. Then, in Section 5 we summarize the structural formulas for Dirac Lie groups obtained in the previous two sections, and use them to describe some examples explicitly. Following this, in Section 6 we relate Dirac Lie groups to the theory of quasi-Poisson geometry [2] and the language of quasi-Lie bialgebroids [35, 17, 32]. Finally, in Section 7 we study those Dirac Lie groups for which the underlying Courant algebroid is exact.

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1. PRELIMINARIES

We begin with a quick review of Courant algebroids and Dirac structures. A more slow-paced overview and further references can be found in our paper [19].

1.1. Basic definitions. A *Courant algebroid* over a manifold M is a vector bundle $\mathbb{A} \rightarrow M$, together with a bundle map $\mathbf{a}: \mathbb{A} \rightarrow TM$ called the *anchor*, a bundle metric¹ $\langle \cdot, \cdot \rangle$, and a bilinear bracket $[\![\cdot, \cdot]\!]$ on its space of sections $\Gamma(\mathbb{A})$. These are required to satisfy the following axioms, for all sections $x_1, x_2, x_3 \in \Gamma(\mathbb{A})$:

- c1) $[\![x_1, [x_2, x_3]]\!] = [[\![x_1, x_2], x_3] + [x_2, [x_1, x_3]]]$,
- c2) $\mathbf{a}(x_1)\langle x_2, x_3 \rangle = \langle [x_1, x_2], x_3 \rangle + \langle x_2, [x_1, x_3] \rangle$,
- c3) $[x_1, x_2] + [x_2, x_1] = \mathbf{a}^*(d\langle x_1, x_2 \rangle)$.

Here $\mathbf{a}^*: T^*M \rightarrow \mathbb{A}^* \cong \mathbb{A}$ is the dual map to \mathbf{a} . The axioms c1)-c3) imply various other properties, in particular

- c4) $[x_1, f x_2] = f [x_1, x_2] + \mathbf{a}(x_1)(f) x_2$,
- c5) $[f x_1, x_2] = f [x_1, x_2] - \mathbf{a}(x_2)(f) x_1 + \langle x_1, x_2 \rangle \mathbf{a}^*(df)$
- c6) $\mathbf{a}([x_1, x_2]) = [\mathbf{a}(x_1), \mathbf{a}(x_2)]$,

for sections $x_i \in \Gamma(\mathbb{A})$ and functions $f \in C^\infty(M)$. We will refer to the bracket $[\![\cdot, \cdot]\!]$ as the *Courant bracket* (it is also known as the *Dorfman bracket*).

Following Ševera [36], the Courant algebroid is called *exact* if the sequence

$$0 \rightarrow T^*M \rightarrow \mathbb{A} \rightarrow TM \rightarrow 0$$

is exact. In particular, the bundle metric of \mathbb{A} is of split signature, and T^*M is a Lagrangian subbundle. Any choice of a Lagrangian splitting $\mathbf{l}: TM \rightarrow \mathbb{A}$ gives an isomorphism $\mathbb{A} \xrightarrow{\cong} TM \oplus T^*M$, with inverse map $v + \alpha \mapsto \mathbf{l}(v) + \mathbf{a}^*(\alpha)$. Under this identification, the anchor map \mathbf{a} becomes projection to the first summand, the bilinear form is $\langle v_1 + \alpha_1, v_2 + \alpha_2 \rangle = \langle \alpha_2, v_1 \rangle + \langle \alpha_1, v_2 \rangle$, and the Courant bracket reads

$$[\![v_1 + \alpha_1, v_2 + \alpha_2]\!] = [v_1, v_2] + \mathcal{L}_{v_1} \alpha_2 - \iota(v_2) d\alpha_1 + \iota(v_1) \iota(v_2) \eta,$$

for vector fields $v_i \in \mathfrak{X}(M)$ and 1-forms $\alpha_i \in \Omega^1(M)$. Here $\eta \in \Omega^3(M)$ is the closed 3-form obtained as

$$(1) \quad \iota(v_1) \iota(v_2) \iota(v_3) \eta = \langle [\![\mathbf{l}(v_1), \mathbf{l}(v_2)]\!], \mathbf{l}(v_3) \rangle.$$

Conversely, given a closed 3-form η , these formulas define a Courant algebroid structure on $TM \oplus T^*M$. We will denote this Courant algebroid by $\mathbb{T}M_\eta$, or simply $\mathbb{T}M$ if $\eta = 0$. The set of Lagrangian splittings is an affine space, with $\Omega^2(M)$ as its space of motions, and a change of the splitting by a 2-form ω changes η to $\eta + d\omega$.

Another class of examples of Courant algebroids is as follows. Suppose \mathfrak{g} is a Lie algebra equipped with an invariant metric. Given a Lie algebra action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ on a manifold M , let $\mathbb{A} = M \times \mathfrak{g}$ with anchor map $\mathbf{a}(m, \xi) = \varrho(\xi)_m$, and with the bundle metric coming from the metric on \mathfrak{g} . As shown in [19], the Lie bracket on constant sections $\mathfrak{g} \subseteq C^\infty(M, \mathfrak{g}) = \Gamma(\mathbb{A})$ extends to a Courant bracket *if and only if* the stabilizers $\mathfrak{g}_m \subseteq \mathfrak{g}$ are coisotropic,

¹In this paper, we take ‘metric’ to mean a non-degenerate symmetric bilinear form.

i.e. $\mathfrak{g}_m \supseteq \mathfrak{g}_m^\perp$. Explicitly, for $\xi_1, \xi_2 \in \Gamma(\mathbb{A}) = C^\infty(M, \mathfrak{g})$ the Courant bracket reads (see [19, § 4])

$$(2) \quad \llbracket \xi_1, \xi_2 \rrbracket = [\xi_1, \xi_2] + \mathcal{L}_{\varrho(\xi_1)}\xi_2 - \mathcal{L}_{\varrho(\xi_2)}\xi_1 + \varrho^*\langle d\xi_1, \xi_2 \rangle.$$

Here $\varrho^*: T^*M \rightarrow M \times \mathfrak{g}$ is the dual map to the action map $\varrho: M \times \mathfrak{g} \rightarrow TM$, using the metric to identify $\mathfrak{g}^* \cong \mathfrak{g}$. We refer to $M \times \mathfrak{g}$ with bracket (2) as an *action Courant algebroid*.

For any Courant algebroid \mathbb{A} , we denote by $\overline{\mathbb{A}}$ the Courant algebroid with the same bracket and anchor, but with the opposite bilinear form.

1.2. Involutive subbundles. Let $\mathbb{A} \rightarrow M$ be a Courant algebroid. A subbundle $E \subseteq \mathbb{A}$ along a submanifold $S \subseteq M$ is called *involutive* if it has the property

$$x_1|_S, x_2|_S \in \Gamma(E) \Rightarrow \llbracket x_1, x_2 \rrbracket|_S \in \Gamma(E).$$

We stress that this property need not define a bracket on $\Gamma(E)$, in general. Using the properties c4 and c5 of Courant algebroids, one finds that if $E \rightarrow S$ is an involutive subbundle, with $0 < \text{rank}(E) < \text{rank}(\mathbb{A})$, then

$$\mathfrak{a}(E) \subseteq TS, \quad \mathfrak{a}(E^\perp) \subseteq TS.$$

Note that the second property is not preserved under intersections of bundles, and indeed a sub-bundle given as the intersection of involutive sub-bundles need not be involutive (unless these subbundles are defined over the same submanifold). An involutive Lagrangian subbundle $E \subseteq \mathbb{A}$ along $S \subseteq M$ is called a *Dirac structure along S* . For instance, if $\mathbb{A} = TM$ is the standard Courant algebroid, then $T^*M|_S$ and $TS \oplus \text{ann}(TS)$ are Dirac structures along S .

A Dirac structure along $S = M$ is simply called a Dirac structure. Dirac structures were introduced by Courant [6] and Liu-Weinstein-Xu [21]; the notion of a Dirac structure along a submanifold goes back to Ševera [36] and was developed in [4, 5, 32].

1.3. Courant relations. A smooth relation $S: M_0 \dashrightarrow M_1$ between manifolds is an immersed submanifold $S \subseteq M_1 \times M_0$. We will write $m_0 \sim_S m_1$ if $(m_1, m_0) \in S$. Given smooth relations $S: M_0 \dashrightarrow M_1$ and $S': M_1 \dashrightarrow M_2$, the set-theoretic composition $S' \circ S$ is the image of

$$(3) \quad S' \diamond S = (S' \times S) \cap (M_2 \times (M_1)_\Delta \times M_0)$$

under projection to $M_2 \times M_0$. We say that the two relations *compose cleanly* if (3) is a clean intersection in the sense of Bott (i.e. it is smooth, and the intersection of the tangent bundles is the tangent bundle of the intersection), and the map from $S' \diamond S$ to $M_2 \times M_0$ has constant rank. In this case, the composition $S' \circ S: M_0 \dashrightarrow M_2$ is a well-defined smooth relation. See Appendix A for more information on the composition of smooth relations. For background on clean intersections of manifolds, see e.g. [14, page 490].

Specializing to vector bundles, Lie algebroids and Courant algebroids, we define

- Definition 1.1.* (a) A *vector bundle relation* (\mathcal{VB} -relation) $R: V_0 \dashrightarrow V_1$ between vector bundles $V_i \rightarrow M_i$ is a subbundle $R \subseteq V_1 \times V_0$ along a submanifold $S \subseteq M_1 \times M_0$.
 (b) A *Lie algebroid relation* (\mathcal{LA} -relation) $R: E_0 \dashrightarrow E_1$ between Lie algebroids $E_i \rightarrow M_i$ is a Lie subalgebroid $R \subseteq E_1 \times E_0$ along a submanifold $S \subseteq M_1 \times M_0$.

(c) A *Courant relation* (\mathcal{CA} -relation) $R: \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ between Courant algebroids $\mathbb{A}_i \rightarrow M_i$ is a Dirac structure $R \subseteq \mathbb{A}_1 \times \overline{\mathbb{A}_0}$ along a submanifold $S \subseteq M_1 \times M_0$.

For a \mathcal{VB} -relation $R: V_0 \dashrightarrow V_1$, with underlying relation $S: M_0 \dashrightarrow M_1$, we define $\ker(R) \subseteq p_{M_0}^* V_0$, $\text{ran}(R) \subseteq p_{M_1}^* V_1$ to be the kernel and range of the bundle map $R \rightarrow p_{M_1}^* V_1$, $(v_1, v_0) \mapsto v_1$ (where $p_{M_i}: S \rightarrow M_i$, $(m_1, m_0) \mapsto m_i$).

We will sometimes depict \mathcal{VB} -relations by diagrams as follows:

$$\begin{array}{ccc} V_0 & \xrightarrow{R} & V_1 \\ \downarrow & & \downarrow \\ M_0 & \xrightarrow{S} & M_1 \end{array}$$

The dashed arrow \dashrightarrow for S is replaced by a solid arrow if S is the graph of a map, and similarly for R . Given a \mathcal{VB} -relation $R: V_0 \dashrightarrow V_1$ and sections $\sigma_i \in \Gamma(V_i)$, we will write $\sigma_0 \sim_R \sigma_1$ if (σ_1, σ_0) restricts to a section of R . Given a relation $S: M_0 \rightarrow M_1$ and functions $f_i \in C^\infty(M_i)$, we write $f_0 \sim_S f_1$ if $f_0(m_0) = f_1(m_1)$ for all $(m_1, m_0) \in S$. The following is clear from the definitions:

Proposition 1.2. *Suppose $\mathbb{A}_0, \mathbb{A}_1$ are Courant algebroids and $R: \mathbb{A}_0 \rightarrow \mathbb{A}_1$ is a \mathcal{VB} -relation with underlying relation $S: M_0 \dashrightarrow M_1$. Suppose $\sigma_0 \sim_R \sigma_1$ and $\tau_0 \sim_R \tau_1$. Then*

- (a) *If R is Lagrangian, $\langle \sigma_0, \tau_0 \rangle \sim_S \langle \sigma_1, \tau_1 \rangle$.*
- (b) *If R is involutive, $[[\sigma_0, \tau_0]] \sim_R [[\sigma_1, \tau_1]]$.*

The composition $R' \circ R$ of two \mathcal{VB} -relations is called *clean* if it is clean as a composition of submanifolds. It is then automatic that $R' \diamond R$ and $R' \circ R$ are smooth subbundles along $S' \diamond S$ and $S' \circ S$, respectively, where S', S are the base manifolds of R', R . Conversely, if the base manifolds compose cleanly, and the pointwise fibers of $R' \diamond R$, $R' \circ R$ have constant rank, then the subbundles compose cleanly.

Remark 1.3. Here it is convenient to work with the characterization of vector bundles and their morphisms in terms of scalar multiplication, due to Grabowski and Rotkiewicz [10]. Specifically, a smooth submanifold of a vector bundle is a vector subbundle if and only if it is invariant under scalar multiplication [10, Theorem 2.3], and a smooth map between vector bundles is a vector bundle homomorphism if and only if it intertwines scalar multiplication [10, Theorem 2.4].

The following proposition shows that the clean composition of \mathcal{CA} -relations is again a \mathcal{CA} -relation. There is a parallel statement for \mathcal{LA} -relations, with a similar proof.

Proposition 1.4. *Suppose $\mathbb{A}_i \rightarrow M_i$ are Courant algebroids, and that $R: \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ and $R': \mathbb{A}_1 \dashrightarrow \mathbb{A}_2$ are \mathcal{VB} -relations with clean composition.*

- (a) *If R, R' are involutive then so is $R' \circ R$.*
- (b) *If R, R' are Lagrangian then so is $R' \circ R$.*

In particular, if R, R' are Courant relations then so is $R' \circ R$.

Proof. (a) Let $p: M_2 \times M_1 \times M_1 \times M_0 \rightarrow M_2 \times M_0$ be the projection, and let

$$Q: \mathbb{A}_2 \times \overline{\mathbb{A}_1} \times \mathbb{A}_1 \times \overline{\mathbb{A}_0} \dashrightarrow \mathbb{A}_2 \times \mathbb{A}_0$$

be the relation defined by $(\mathbb{A}_2)_\Delta \times (\mathbb{A}_1)_\Delta \times (\mathbb{A}_0)_\Delta$. Under this relation, $\tilde{\sigma} \sim_Q \sigma$ if and only if the restriction of $\tilde{\sigma} - p^*\sigma$ to $M_2 \times (M_1)_\Delta \times M_0$ takes values in $0 \times (\mathbb{A}_1)_\Delta \times 0$. Since Q is involutive, we have

$$\tilde{\sigma} \sim_Q \sigma, \tilde{\tau} \sim_Q \tau \Rightarrow [\tilde{\sigma}, \tilde{\tau}] \sim_Q [\sigma, \tau].$$

Suppose R, R' are involutive. Let σ be a section of $\mathbb{A}_2 \times \overline{\mathbb{A}_0}$ whose restriction to $S' \circ S$ takes values in $R' \circ R$. Since $R' \diamond R \rightarrow R' \circ R$ is a surjective vector bundle homomorphism covering a submersion $S' \diamond S \rightarrow S' \circ S$, the restriction $\sigma|_{S' \circ S}$ admits a lift to a section $\tilde{\sigma}|_{S' \diamond S}$ of $R' \diamond R$. By definition, $\tilde{\sigma}|_{S' \diamond S} - p^*\sigma|_{S' \diamond S}$ takes values in $0 \times (\mathbb{A}_1)_\Delta \times 0$. Since the bundles $R' \times R \rightarrow S' \times S$ and $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \mathbb{A}_0 \rightarrow M_2 \times (M_1)_\Delta \times M_0$ intersect cleanly, we may choose $\tilde{\sigma} \in \Gamma(\mathbb{A}_2 \times \overline{\mathbb{A}_1} \times \mathbb{A}_1 \times \overline{\mathbb{A}_0})$ such that

- (i) $\tilde{\sigma}|_{S' \diamond S} = \tilde{\sigma}|_{S' \circ S}$,
- (ii) $\tilde{\sigma}|_{S' \times S}$ takes values in $R' \times R$,
- (iii) $(\tilde{\sigma} - p^*\sigma)|_{M_2 \times (M_1)_\Delta \times M_0}$ takes values in $0 \times (\mathbb{A}_1)_\Delta \times 0$, i.e. $\tilde{\sigma} \sim_Q \sigma$.

Note that (iii) implies that $\tilde{\sigma}|_{M_2 \times (M_1)_\Delta \times M_0}$ takes values in $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \overline{\mathbb{A}_0}$.

Given another section τ of $\mathbb{A}_2 \times \overline{\mathbb{A}_0}$ whose restriction $\tau|_{S' \circ S}$ takes values in $R' \circ R$, let $\tilde{\tau}$ be constructed similarly. Since $R' \times R$ and $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \overline{\mathbb{A}_0}$ are involutive, the restriction of $[\tilde{\sigma}, \tilde{\tau}]$ to $S' \times S$ takes values in $R' \times R$, while the restriction to $M_2 \times (M_1)_\Delta \times M_0$ takes values in $\mathbb{A}_2 \times (\mathbb{A}_1)_\Delta \times \overline{\mathbb{A}_0}$. Hence $[\tilde{\sigma}, \tilde{\tau}]|_{S' \diamond S}$ takes values in $R' \diamond R$. Since $[\tilde{\sigma}, \tilde{\tau}] \sim_Q [\sigma, \tau]$, this shows that $[\sigma, \tau]|_{S' \circ S}$ takes values in $R' \circ R$.

Part (b) follows from the well-known statement that the composition of Lagrangian relations of vector spaces is again Lagrangian (Lemma A.1). \square

A *Courant morphism* [36] is a Courant relation $R: \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ such that the underlying relation $S: M_0 \dashrightarrow M_1$ is the graph of a map $\Phi: M_0 \rightarrow M_1$. (In contrast with vector bundle morphisms or Lie algebroid morphisms, one does not require that R be a graph.) As a special case of Proposition 1.4, the composition of Courant morphisms is again a Courant morphism.

Example 1.5. Any smooth map $\Phi: M_0 \rightarrow M_1$ has a *standard lift* to a Courant morphism $R_\Phi: \mathbb{T}M_0 \dashrightarrow \mathbb{T}M_1$, given by

$$(4) \quad v_0 + \alpha_0 \sim_{R_\Phi} v_1 + \alpha_1 \quad \Leftrightarrow \quad v_1 = T\Phi(v_0), \text{ and } \alpha_0 = T\Phi^*\alpha_1.$$

More generally, suppose $\eta_i \in \Omega^3(M_i)$ are closed three forms, and $\omega \in \Omega^2(M_0)$ with $\eta_0 = \Phi^*\eta_1 + d\omega$. Then there is a Courant morphism $R_{\Phi, \omega}: (\mathbb{T}M_0)_{\eta_0} \dashrightarrow (\mathbb{T}M_1)_{\eta_1}$ given by [5, Example 2.11]

$$v_0 + \alpha_0 \sim_{R_{\Phi, \omega}} v_1 + \alpha_1 \quad \Leftrightarrow \quad v_1 = T\Phi(v_0), \text{ and } \alpha_0 = T\Phi^*\alpha_1 - \iota(v_0)\omega.$$

1.4. Manin pairs. A *Manin pair* (\mathbb{A}, E) is a Courant algebroid $\mathbb{A} \rightarrow M$ together with a Dirac structure $E \subseteq \mathbb{A}$. If $M = \text{pt}$, this reduces to the usual notion of a Manin pair of Lie algebras. A *morphism of Manin pairs* [5]

$$R: (\mathbb{A}, E) \dashrightarrow (\mathbb{A}', E'),$$

with underlying map $\Phi: M \rightarrow M'$, is a morphism of Courant algebroids with the property that for all $m \in M$, any element of $E'_{\Phi(m)}$ is R -related to a unique element of E_m .

Equivalently, in terms of composition of relations,

$$\Phi^*E' = R \circ E, \quad \ker(R) \cap E = 0.$$

One obtains a bundle map $\Phi^*E' \rightarrow E$, associating to each $x' \in E'_{\Phi(m)}$ the unique $x \in E_m$ to which it is R -related. This bundle map is a comorphism of Lie algebroids [24], thus in particular the map $\Phi^*: \Gamma(E') \rightarrow \Gamma(E)$ preserves Lie brackets.

Example 1.6. For any Manin pair (\mathbb{A}, E) over M , there is a morphism of Manin pairs

$$R: (\mathbb{T}M, TM) \dashrightarrow (\mathbb{A}, E)$$

where $v + \alpha \sim_R x$ if and only if $v = \mathbf{a}(x)$ and $x - \mathbf{a}^*(\alpha) \in E$.

Example 1.7. Suppose M, M' are Poisson manifolds with bivector fields π, π' . Let $\Phi: M \rightarrow M'$ be a smooth map. Then the standard lift $R_\Phi: \mathbb{T}M \dashrightarrow \mathbb{T}M'$ (cf. (4)) defines a morphism of Manin pairs $R_\Phi: (\mathbb{T}M, \text{Gr}_\pi) \dashrightarrow (\mathbb{T}M', \text{Gr}_{\pi'})$ if and only if Φ is a Poisson map.

2. DIRAC LIE GROUPS

The definition of Dirac Lie group structures (Definition 2.5 below) requires that the ambient Courant algebroid itself be multiplicative, in the sense that it carries the structure of a \mathcal{CA} -groupoid.

2.1. \mathcal{CA} -groupoids. For any groupoid $H \rightrightarrows H^{(0)}$ and $k > 0$ denote $H^{(k)} = \{(g_1, \dots, g_k) \in H^k \mid s(g_i) = t(g_{i+1}), i = 1, \dots, k-1\}$. Let $\text{Mult}_H: H^{(2)} \rightarrow H$, $(X, Y) \rightarrow X \circ Y$ denote the groupoid multiplication, and

$$\text{gr}(\text{Mult}_H) = \{(X \circ Y, X, Y) \mid (X, Y) \in H^{(2)}\} \subseteq H^3$$

its graph.

Definition 2.1. Let $H \rightrightarrows H^{(0)}$ be a Lie groupoid.

- (a) A \mathcal{VB} -groupoid over H is a vector bundle $V \rightarrow H$, equipped with a groupoid structure such that $\text{gr}(\text{Mult}_V) \subseteq V^3$ is a vector subbundle along $\text{gr}(\text{Mult}_H)$.
- (b) An \mathcal{LA} -groupoid over H is a Lie algebroid $E \rightarrow H$, equipped with a groupoid structure such that $\text{gr}(\text{Mult}_E) \subseteq E^3$ is a Lie subalgebroid along $\text{gr}(\text{Mult}_H)$.
- (c) A \mathcal{CA} -groupoid over H is a Courant algebroid $\mathbb{A} \rightarrow H$, equipped with a groupoid structure such that $\text{gr}(\text{Mult}_\mathbb{A}) \subseteq \mathbb{A} \times \overline{\mathbb{A}} \times \overline{\mathbb{A}}$ is a Dirac structure along $\text{gr}(\text{Mult}_H)$.

In other words, we require that the groupoid multiplication is a \mathcal{VB} -relation, \mathcal{LA} -relation or \mathcal{CA} -relation, respectively. It is common to indicate a \mathcal{VB} -groupoid V by a diagram

$$\begin{array}{ccc} V & \rightrightarrows & V^{(0)} \\ \downarrow & & \downarrow \\ H & \rightrightarrows & H^{(0)} \end{array}$$

Remark 2.2. (a) The definition of \mathcal{VB} -groupoids given above is somewhat shorter than Pradines' original definition [33], which requires that all the groupoid structure maps of V are morphisms of vector bundles. The equivalence of the two definitions follows from Grabowski-Rotkiewicz's Remark 1.3. For instance, since $V^{(0)} \subseteq V$ is a smooth

submanifold invariant under scalar multiplication, it is a vector subbundle. Similarly, since $s_V, t_V: V \rightarrow V^{(0)}$ are smooth maps intertwining scalar multiplication they are vector bundle morphisms.

- (b) \mathcal{LA} -groupoids are due to Mackenzie [22, 23]. The definition above implies that $V^{(0)}$ is a Lie subalgebroid along $H^{(0)}$, and that all the groupoid structure maps are morphisms of Lie algebroids.
- (c) The concept of a \mathcal{CA} -groupoid (also called Courant groupoid) was suggested by Mehta [27, Example 3.8] and Ortiz [31, Section 7.3], and developed in detail in [20].

A relation of \mathcal{CA} -groupoids $R: \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ is a \mathcal{CA} -relation such that $R \subseteq \mathbb{A}_1 \times \overline{\mathbb{A}_0}$ is a Lie subgroupoid. If the underlying relation $S: H_0 \dashrightarrow H_1$ is the graph of a groupoid homomorphism, then R is called a *morphism of \mathcal{CA} -groupoids*. Relations and morphisms of $\mathcal{VB}, \mathcal{LA}$ -groupoids are defined similarly.

Proposition 2.3. *Let $\mathbb{A} \rightarrow H$ be a \mathcal{CA} -groupoid. Then the set of units $\mathbb{A}^{(0)} \subseteq \mathbb{A}$ is a Dirac structure along $H^{(0)} \subseteq H$. Furthermore, the groupoid inversion defines a morphism of Courant algebroids $\text{Inv}_{\mathbb{A}}: \mathbb{A} \dashrightarrow \overline{\mathbb{A}}$ over $\text{Inv}_H: H \rightarrow H$.*

Proof. Define a relation $D: \overline{\mathbb{A}} \times \mathbb{A} \dashrightarrow \mathbb{A}$, where $(x_1, x_2) \sim_D x$ if and only if $x = x_1^{-1} \circ x_2$. Since

$$D = \{(x_1^{-1} \circ x_2, x_1, x_2) \mid t(x_1) = t(x_2)\} \subseteq \mathbb{A} \times \mathbb{A} \times \overline{\mathbb{A}}.$$

is obtained from $\text{gr}(\text{Mult}_{\mathbb{A}})$ by a cyclic permutation of components (and an overall sign change of the metric), it is a Dirac structure along the graph of the relation $H \times H \dashrightarrow H$, $(g_1, g_2) \sim g_1^{-1} g_2$. On the other hand, we may think of the diagonal in $\overline{\mathbb{A}} \times \mathbb{A}$ as a Courant relation $\mathbb{A}_{\Delta}: 0 \dashrightarrow \overline{\mathbb{A}} \times \mathbb{A}$, with underlying relation $H_{\Delta}: \text{pt} \dashrightarrow H \times H$. Observe $\mathbb{A}^{(0)} = D \circ \mathbb{A}_{\Delta}$, where the composition is clean. Hence $\mathbb{A}^{(0)}$ is a Dirac structure along $H^{(0)}$. Similarly, the graph of the groupoid inversion $\text{gr}(\text{Inv}_{\mathbb{A}}) \subseteq \overline{\mathbb{A}} \times \overline{\mathbb{A}}$ is a clean composition of Courant relations $\text{gr}(\text{Inv}_{\mathbb{A}}) = \mathbb{A}^{(0)} \circ \text{gr}(\text{Mult}_{\mathbb{A}})$. \square

Note that D and \mathbb{A}_{Δ} are relations of Courant groupoids, if we take $\overline{\mathbb{A}} \times \mathbb{A}$ with the pair groupoid structure.

2.2. Multiplicative Manin pairs and Dirac Lie group structures.

Definition 2.4. [20, 31, 5, 27] A *multiplicative Manin pair* is a Manin pair (\mathbb{A}, E) , where $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ is a \mathcal{CA} -groupoid over $H \rightrightarrows H^{(0)}$, and $E \rightrightarrows E^{(0)}$ is a \mathcal{VB} -subgroupoid of \mathbb{A} . A *morphism of multiplicative Manin pairs* $R: (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$ is a morphism of Manin pairs which is also a morphism of \mathcal{CA} -groupoids $R: \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$.

The involutivity condition implies that E inherits the structure of an \mathcal{LA} -groupoid.

As shown in Proposition 2.3, for any \mathcal{CA} -groupoid structure $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$, the space $\mathbb{A}^{(0)}$ of units is a Dirac structure along $H^{(0)}$. In this paper, we are mainly concerned with the case that $H^{(0)} = \text{pt}$, such that H is a group. In this case, the groupoid multiplication defines a

Courant morphism $R = \text{gr}(\text{Mult}_{\mathbb{A}})$ covering the group multiplication,

$$\begin{array}{ccc} \mathbb{A} \times \mathbb{A} & \xrightarrow{R} & \mathbb{A} \\ \downarrow & & \downarrow \\ H \times H & \xrightarrow{\text{Mult}_H} & H \end{array}$$

Definition 2.5. A *Dirac Lie group structure* on a Lie group H is a multiplicative Manin pair (\mathbb{A}, E) over H such that the multiplication morphism $\text{gr}(\text{Mult}_{\mathbb{A}}): (\mathbb{A}, E) \times (\mathbb{A}, E) \dashrightarrow (\mathbb{A}, E)$ is a morphism of Manin pairs.

Given Dirac Lie group structures $(\mathbb{A}, E), (\mathbb{A}', E')$ on Lie groups H, H' , a morphism of multiplicative Manin pairs $(\mathbb{A}, E) \dashrightarrow (\mathbb{A}', E')$ is called a *morphism of Dirac Lie groups*.

Remark 2.6. In other words, we define the category of Dirac Lie groups to be the subcategory of *group like* objects in the category of Manin pairs. Meanwhile, the category of multiplicative Manin pairs is the subcategory of *groupoid like* objects in the category of Manin pairs. Our definition is more restrictive than that of Ortiz [31, 30] and Jotz [15], where Dirac Lie group structures are taken to be arbitrary multiplicative Manin pairs over H . Note that [15, 30] only explore the case $\mathbb{A} = \mathbb{T}H$. In an earlier paper, Milburn [28] gives a ‘categorical’ definition of what he calls *Dirac groups*, similar to the Ortiz-Jotz definition.

Proposition 2.7. A multiplicative Manin pair (\mathbb{A}, E) defines a Dirac Lie group structure if and only if E is a wide subgroupoid of \mathbb{A} , i.e. $\mathbb{A}^{(0)} = E^{(0)}$.

Proof. Suppose (\mathbb{A}, E) is a multiplicative Manin pair over H , and that $\mathbb{A}^{(0)} = E^{(0)}$. We will show that the multiplication morphism is a morphism of Manin pairs.

By Proposition 2.3, $\mathfrak{g} = \mathbb{A}^{(0)}$ is Lagrangian, as is E_e . Since E contains the units, it follows that $E_e = \mathfrak{g}$. More generally, for any $h \in H$ the source and target map give isomorphisms $s_h, t_h: E_h \rightarrow \mathfrak{g}$. Hence if $h_1, h_2 \in H$ are given, then any $x \in E_{h_1 h_2}$ can be uniquely written as a product $x = x_1 \circ x_2$ with $x_i \in E_{h_i}$: x_1 is uniquely determined by $t(x_1) = t(x)$, and then $x_2 = x_1^{-1} \circ x$. This shows that $\text{Mult}_{\mathbb{A}}$ gives a morphism of Manin pairs.

Conversely, suppose $E^{(0)}$ is a proper subspace of $A^{(0)}$. Then $\dim E^{(0)} < \text{rank}(E) = \dim A^{(0)}$. In particular, $\ker(s|_E)$ is non-trivial. Since

$$\{(x^{-1}, x) \mid x \in \ker(s|_E)\} \subseteq \ker(\text{Mult}_{\mathbb{A}}) \cap (E \times E),$$

this shows that $\text{Mult}_{\mathbb{A}}$ does not define a morphism of Manin pairs. \square

2.3. Examples.

Example 2.8. For any Courant algebroid $\mathbb{A} \rightarrow M$, the direct product $\overline{\mathbb{A}} \times \mathbb{A} \rightarrow M \times M$, with groupoid structure that of a pair groupoid, defines a \mathcal{CA} -groupoid structure over the pair groupoid $M \times M \rightrightarrows M$:

$$\begin{array}{ccc} \overline{\mathbb{A}} \times \mathbb{A} & \rightrightarrows & \mathbb{A} \\ \downarrow & & \downarrow \\ M \times M & \rightrightarrows & M \end{array}$$

If (\mathbb{A}, E) is a Manin pair, then $(\overline{\mathbb{A}} \times \mathbb{A}, E \times E)$ becomes a multiplicative Manin pair. If $M = \text{pt}$, so that $\mathbb{A} = \mathfrak{g}$ is a quadratic Lie algebra, the diagonal $\mathfrak{g}_\Delta \subseteq \overline{\mathfrak{g}} \oplus \mathfrak{g}$ defines a Dirac Lie group structure on $H = \{e\}$.

Example 2.9. The standard Courant algebroid over any Lie groupoid $H \rightrightarrows H^{(0)}$ is a \mathcal{CA} -groupoid $\mathbb{T}H \rightrightarrows TH^{(0)} \oplus A^*H$, where $AH \rightarrow H^{(0)}$ is the Lie algebroid of H , and A^*H its dual. The \mathcal{VB} -groupoid structure is given as the direct sum of the tangent prolongation $TH \rightrightarrows TH^{(0)}$ and the cotangent groupoid $T^*H \rightrightarrows A^*H$. See [27, Example 3.8] and [20, Example 9]. Both $(\mathbb{T}H, T^*H)$ and $(\mathbb{T}H, TH)$ are multiplicative Manin pairs.

In particular, if H is a Lie group, the \mathcal{VB} -groupoid structure on $\mathbb{T}H$ is the direct product of the group $TH \rightrightarrows \text{pt}$ with the symplectic groupoid $T^*H \rightrightarrows \mathfrak{h}^*$:

$$\begin{array}{ccc} \mathbb{T}H & \rightrightarrows & \mathfrak{h}^* \\ \downarrow & & \downarrow \\ H & \rightrightarrows & \text{pt} \end{array}$$

If $(\mathbb{T}H, E)$ is a Dirac Lie group structure, then $E \cap TH = 0$ since the source and target maps $E \rightarrow \mathfrak{h}^*$ are surjective. Thus E is the graph of a bivector field $\pi \in \Gamma(\wedge^2 TH)$. The condition that E is a subgroupoid translates into the condition that π is multiplicative, i.e. a Poisson-Lie group structure. In fact the following was obtained by Ortiz [30] and Jotz [15], as part of a general classification of multiplicative Manin pairs for $\mathbb{A} = \mathbb{T}H$:

Proposition 2.10. *The Dirac Lie group structures for the standard Courant algebroid over a Lie group H are exactly those of the form $(\mathbb{T}H, \text{Gr}_\pi)$ where π defines a Poisson-Lie group structure on H . If $(H, \pi), (H', \pi')$ are Poisson Lie groups and $\Phi: H \rightarrow H'$ is a Lie group homomorphism, then the standard lift of Φ is a Dirac Lie group morphism if and only if Φ is a Poisson Lie group morphism, i.e. $\pi \sim_\Phi \pi'$.*

As a special case, any Lie group carries a ‘trivial’ Dirac Lie group structure $(\mathbb{T}H, T^*H)$. The Manin pair $(\mathbb{T}H, TH)$ is multiplicative, but is not a Dirac Lie group structure in our sense since TH is not a wide subgroupoid.

Example 2.11. For any multiplicative Manin pair (\mathbb{A}, E) , the morphism $(\mathbb{T}H, TH) \dashrightarrow (\mathbb{A}, E)$ (cf. Example 1.6) is a morphism of multiplicative Manin pairs.

Example 2.12. [1, § 3.4] Let G be a Lie group whose Lie algebra \mathfrak{g} carries an invariant metric B . Then there is a \mathcal{CA} -groupoid structure on G ,

$$\begin{array}{ccc} G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g}) & \rightrightarrows & \mathfrak{g} \\ \downarrow & & \downarrow \\ G & \rightrightarrows & \text{pt} \end{array}$$

Here the \mathcal{VB} -groupoid structure is the direct product of the group $G \rightrightarrows \text{pt}$ with the pair groupoid $\overline{\mathfrak{g}} \oplus \mathfrak{g} \rightrightarrows \mathfrak{g}$. As a Courant algebroid, $\mathbb{A} = G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g})$ is the action Courant algebroid for the following action of $\overline{\mathfrak{g}} \oplus \mathfrak{g}$ on G

$$\varrho(\zeta_1, \zeta_2) = \zeta_2^L - \zeta_1^R$$

where ζ^L, ζ^R are the left-,right-invariant vector fields defined by $\zeta \in \mathfrak{g}$. Since the action ϱ is transitive, the Courant algebroid \mathbb{A} is exact. In fact there is an explicit isomorphism of Courant algebroids $\kappa: G \times (\overline{\mathfrak{g}} \oplus \mathfrak{g}) \rightarrow \mathbb{T}G_\eta$, where $\eta = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)$ is the Cartan 3-form:

$$(5) \quad \kappa(\zeta_1, \zeta_2) = \left(\zeta_2^L - \zeta_1^R, \frac{1}{2}B(\theta^L, \zeta_2) + \frac{1}{2}B(\theta^R, \zeta_1) \right).$$

Here, $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ are the left-,right-invariant Maurer-Cartan forms, defined by the property

$$\iota(\zeta^L)\theta^L = \zeta = \iota(\zeta^R)\theta^R, \quad \zeta \in \mathfrak{g}.$$

The subbundle $E = G \times \mathfrak{g}_\Delta$ defines a Dirac Lie group structure on G . This is the *Cartan-Dirac structure* on G , found independently by Alekseev, Ševera and Strobl. Its multiplicative properties were noted in [1].

2.4. Constructions with \mathcal{CA} -groupoids. In this section we will collect some further properties and constructions for \mathcal{CA} -groupoids. While we are mainly interested in the case $H^{(0)} = \text{pt}$, the general proofs are more conceptual and in any case not harder.

2.4.1. Basic properties. Given a Lie groupoid $H \rightrightarrows H^{(0)}$, let $TH \rightrightarrows TH^{(0)}$ be its tangent prolongation and $T^*H \rightrightarrows A^*H$ the cotangent groupoid.

Proposition 2.13. *For any \mathcal{CA} -groupoid $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ over $H \rightrightarrows H^{(0)}$, the anchor map defines a morphism of \mathcal{VB} -groupoids $\mathbf{a}: \mathbb{A} \rightarrow TH$, while \mathbf{a}^* defines a morphism of \mathcal{VB} -groupoids $\mathbf{a}^*: T^*H \rightarrow \mathbb{A}^*$.*

Proof. By definition of a \mathcal{CA} -groupoid, the image of $\text{gr}(\text{Mult}_\mathbb{A})$ under the anchor map lies in $T \text{gr}(\text{Mult}_H) = \text{gr}(\text{Mult}_{TH})$. Hence, the graph of \mathbf{a} is a \mathcal{VB} -subgroupoid of $TH \times \mathbb{A}$, proving that \mathbf{a} is a \mathcal{VB} -groupoid homomorphism. Dualizing, $\mathbf{a}^*: T^*H \rightarrow \mathbb{A}^*$ is a \mathcal{VB} -groupoid homomorphism. But the isomorphism $\mathbb{A}^* \cong \mathbb{A}$ given by the metric is an isomorphism of \mathcal{VB} -groupoids. \square

Corollary 2.14. *For any \mathcal{CA} -groupoid \mathbb{A} over H , the diagonal morphism $\mathbb{T}H \dashrightarrow \overline{\mathbb{A}} \times \mathbb{A}$ given by*

$$v + \alpha \sim (x, y) \Leftrightarrow v = \mathbf{a}(x), \quad y - x = \mathbf{a}^*(\alpha)$$

is a morphism of \mathcal{CA} -groupoids.

Proof. As shown in [19, Proposition 1.6], the diagonal morphism is a morphism of Courant algebroids. By Proposition 2.13, it is also a morphism of \mathcal{VB} -groupoids. \square

2.4.2. Reduction and Pull-backs.

Proposition 2.15 (Coisotropic reduction). *Let $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ be a \mathcal{CA} -groupoid over $H \rightrightarrows H^{(0)}$, and let $C \subseteq \mathbb{A}$ be a \mathcal{VB} -subgroupoid along a subgroupoid $K \subseteq H$. Assume that*

- (a) C is co-isotropic,
- (b) C is involutive,
- (c) $\mathbf{a}(C) \subseteq TK, \mathbf{a}(C^\perp) = 0$.

Then the quotient $\mathbb{A}_C = C/C^\perp$ defines a \mathcal{CA} -groupoid structure on K , in such a way that the inclusion map $K \rightarrow H$ lifts to a morphism of \mathcal{CA} -groupoids, $C/C^\perp \dashrightarrow \mathbb{A}$. If $E \subseteq \mathbb{A}$ defines a multiplicative Manin pair (\mathbb{A}, E) , and E is transverse to C then

$$(\mathbb{A}_C, E_C) = (C/C^\perp, (E \cap C)/(E \cap C^\perp))$$

is again a multiplicative Manin pair.

Here transversality means $E|_K + C = \mathbb{A}|_K$, or equivalently $E \cap C^\perp = 0$.

Proof. Since C is a co-isotropic \mathcal{VB} -subgroupoid of \mathbb{A} , C^\perp is a \mathcal{VB} -subgroupoid of C , and $\mathbb{A}_C = C/C^\perp$ inherits a \mathcal{VB} -groupoid structure (see Corollary C.5 from Appendix C for details). By [19, Proposition 2.1], the Courant bracket on \mathbb{A} descends to a Courant bracket on the quotient \mathbb{A}_C , in such a way that

$$(6) \quad S = \{(x, [x]) \mid x \in C\} \subseteq \mathbb{A} \times \overline{\mathbb{A}}_C$$

is a Courant morphism $S: \mathbb{A}_C \dashrightarrow \mathbb{A}$. Here $[x] \in \mathbb{B}$ denotes the image of $x \in C$. The graph of the groupoid multiplication of \mathbb{A}_C is a transverse composition of Courant relations,

$$\text{gr}(\text{Mult}_{\mathbb{A}_C}) = \text{gr}(\text{Mult}_{\mathbb{A}}) \circ (S \times S \times S),$$

hence it is itself a Courant relation. Thus \mathbb{A}_C carries a \mathcal{CA} -groupoid structure. Since S is a Dirac structure along the graph of the inclusion, and also a subgroupoid, it defines a \mathcal{CA} -groupoid morphism.

If $E \subseteq \mathbb{A}$ is a multiplicative Dirac structure transverse to C , then $E_C = E \circ S$ is a transverse composition, and is a multiplicative Dirac structure in \mathbb{A}_C . \square

Proposition 2.16 (Pull-backs). *Let $\mathbb{A} \rightrightarrows \mathbb{A}^{(0)}$ be a \mathcal{CA} -groupoid over $H \rightrightarrows H^{(0)}$, and $\Phi: K \rightarrow H$ a homomorphism of Lie groupoids. Suppose that Φ is transverse to the anchor map $\mathbf{a}: \mathbb{A} \rightarrow TH$. Then the pull-back Courant algebroid $\Phi^!\mathbb{A} \rightrightarrows \Phi^*\mathbb{A}^{(0)}$ inherits the structure of a \mathcal{CA} -groupoid over $K \rightrightarrows K^{(0)}$.*

Proof. By definition (see [19, Proposition 2.7]), the pull-back Courant algebroid is a reduction $\Phi^!\mathbb{A} = (\mathbb{A} \times_{TK})_C$ relative to the coisotropic subbundle C along $\text{gr}(\Phi) \cong K$,

$$C = \mathbb{A} \times_{TH} TK \subseteq \mathbb{A} \times TK,$$

the fiber product relative to the maps $\mathbf{a}_{\mathbb{A}}: \mathbb{A} \rightarrow TH$ and $d\Phi \circ \mathbf{a}_{TK}: TK \rightarrow TH$. Proposition C.1 shows that C is a Lie groupoid. Its space of units $C^{(0)} = \mathbb{A}^{(0)} \times_{TH^{(0)}} A^*K$ is a smooth subbundle of $\mathbb{A}^{(0)} \times A^*K$ along $\text{gr}(\Phi|_{K^{(0)}}) \cong K^{(0)}$. Corollary C.5 from Appendix C shows that $C^\perp \subseteq C$ is a subgroupoid. Hence C/C^\perp inherits a \mathcal{CA} -groupoid structure. \square

3. CLASSIFICATION OF DIRAC LIE GROUP STRUCTURES

In this Section we will give the general classification and construction of Dirac Lie group structures over Lie groups H . The classification will be given in terms of H -equivariant Dirac Manin triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$.

3.1. Vacant \mathcal{LA} -groupoids. Following Mackenzie [22], a \mathcal{VB} -groupoid $V \rightarrow H$ will be called *vacant* if it has the property $V^{(0)} = V|_{H^{(0)}}$.

Lemma 3.1. *For any Dirac Lie group structure (\mathbb{A}, E) over a group H , the sub-bundle E is a vacant \mathcal{LA} -groupoid.*

Proof. The Lie algebroid bracket is induced from the Courant bracket on \mathbb{A} . Since $E^{(0)} \cong \mathbb{A}^{(0)}$ is a Lagrangian subspace of \mathbb{A}_e , it must coincide with E_e . \square

As shown by Mackenzie [22], vacant \mathcal{LA} -groupoids over groups are characterized in terms of Lie-theoretic data. We will review his theory from a mildly different perspective; further details are given in Appendix B.

Definition 3.2. Let H be a Lie group with Lie algebra \mathfrak{h} . A *Lie algebra triple* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Lie algebra \mathfrak{d} with a vector space decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ into two Lie subalgebras $\mathfrak{g}, \mathfrak{h}$. Given an action of H on \mathfrak{d} by automorphisms, which integrates the adjoint action of $\mathfrak{h} \subseteq \mathfrak{d}$ and restricts to the adjoint action of H on \mathfrak{h} , we refer to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ as an *H -equivariant Lie algebra triple*.

We will simply write $h \mapsto \text{Ad}_h$ for the action of H on \mathfrak{d} . Part (b) of the following Proposition associates a vacant \mathcal{LA} -groupoid $E \rightarrow H$ to any H -equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. It is realized as a \mathcal{LA} -subgroupoid of the direct product of $TH \rightrightarrows 0$ with the pair groupoid $\mathfrak{g} \oplus \mathfrak{g} \rightrightarrows \mathfrak{g}$.

Proposition 3.3. *Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be an H -equivariant Lie algebra triple.*

(a) *The subset*

$$V = \{(v, X, X') \mid v \in T_h H, X, X' \in \mathfrak{d}, \text{Ad}_h X' - X = \iota(v)\theta_h^R\},$$

is an \mathcal{LA} -subgroupoid of $TH \times (\mathfrak{d} \oplus \mathfrak{d}) \rightrightarrows \mathfrak{d}$, of rank equal to $\dim \mathfrak{g} + 2 \dim \mathfrak{h}$. Its object space is $V^{(0)} = \mathfrak{d}$.

(b) *The subset*

$$E = \{(v, \xi, \xi') \mid v \in T_h H, \xi, \xi' \in \mathfrak{g}, \text{Ad}_h \xi' - \xi = \iota(v)\theta_h^R\}$$

is a vacant \mathcal{LA} -subgroupoid of $TH \times (\mathfrak{g} \oplus \mathfrak{g}) \rightrightarrows \mathfrak{g}$, of rank equal to $\dim \mathfrak{g}$. Its object space is $E^{(0)} = \mathfrak{g}$. The source map $(v, \xi, \xi') \mapsto \xi'$ is a trivialization of E , and defines a morphism of Lie algebroids $E \rightarrow \mathfrak{g}$.

Proof. (a) The \mathcal{VB} -groupoid $TH \times (\mathfrak{d} \oplus \mathfrak{d}) \rightrightarrows \mathfrak{d}$ may be regarded as a direct sum of two \mathcal{VB} -subgroupoids $TH \rightrightarrows 0$ and $H \times (\mathfrak{d} \oplus \mathfrak{d}) \rightrightarrows \mathfrak{d}$. Right trivialization $TH \cong \mathfrak{h} \rtimes H$ gives a fiberwise injective group isomorphism

$$(7) \quad TH \rightarrow \mathfrak{d} \rtimes H, \quad v \mapsto (\iota_v \theta_h^R, h)$$

where h is the base point of v , and the semi-direct product is relative to Ad . On the other hand, the map

$$(8) \quad H \times (\mathfrak{d} \oplus \mathfrak{d}) \rightarrow \mathfrak{d} \rtimes H, \quad (h, X, X') \mapsto \text{Ad}_h(X') - X$$

is a fiberwise surjective \mathcal{VB} -groupoid homomorphism. The fibered product of the two maps (7), (8) is equal to V , which is hence a \mathcal{VB} -subgroupoid of rank $\dim \mathfrak{h} + 2 \dim \mathfrak{d} - \dim \mathfrak{d} = \dim \mathfrak{g} + 2 \dim \mathfrak{h}$.

Let $H \times H$ act on H by $(h_1, h_2).h = h_1 h h_2^{-1}$, on TH by the tangent lift of this action, and on $\mathfrak{d} \oplus \mathfrak{d}$ by $(h_1, h_2).(X, X') = (\text{Ad}_{h_1} X, \text{Ad}_{h_2} X')$. We obtain a diagonal action on $TH \times (\mathfrak{d} \oplus \mathfrak{d})$ by \mathcal{LA} -groupoid automorphisms. The maps (7), (8) are equivariant relative to the action $(h_1, h_2).(Y, h) = (\text{Ad}_{h_1} Y, h_1 h h_2^{-1})$ on $\mathfrak{d} \rtimes H$, hence V is $H \times H$ -invariant. To verify that V is a \mathcal{LA} -subgroupoid, it hence suffices to check near the group unit. In particular, we may assume that H is connected and simply connected. Let D be a connected Lie group with Lie algebra \mathfrak{d} , and with the action of $\mathfrak{d} \oplus \mathfrak{d}$ by $(X, X') \mapsto X'^L - X^R$. The corresponding action Lie algebroid embeds as a Lie subalgebroid

$$(9) \quad \{(v, X, X') \mid v = X'^L|_d - X^R|_d\} \subseteq TD \times (\mathfrak{d} \oplus \mathfrak{d})$$

(where $d \in D$ is the base point of $v \in TD$). On a neighborhood of $e \in H$, the group homomorphism $H \rightarrow D$ exponentiating $\mathfrak{h} \rightarrow \mathfrak{d}$ is an embedding, and V is simply the intersection of (9) with $TH \times (\mathfrak{d} \oplus \mathfrak{d})$. In particular, it is a Lie subalgebroid of $TH \times (\mathfrak{d} \oplus \mathfrak{d})$.

(b) The same argument as for V shows that E is a subbundle of rank $\dim \mathfrak{g}$. Since E is the intersection of V with the \mathcal{LA} -subgroupoid $TH \times (\mathfrak{g} \oplus \mathfrak{g})$, it is itself an \mathcal{LA} -subgroupoid. Since E has trivial intersection with the subbundle of elements of the form $(v, \xi, 0)$, the source map $TH \times (\mathfrak{g} \oplus \mathfrak{g}) \rightarrow \mathfrak{g}$, $(v, \xi, \xi') \mapsto \xi'$ defines a trivialization of E . Furthermore, since this projection is a Lie algebroid homomorphism, the same is true for its restriction to E . \square

Proposition 3.4. *There is a 1-1 correspondence between*

- (i) *Vacant \mathcal{LA} -groupoids $E \rightrightarrows \mathfrak{g}$ over groups $H \rightrightarrows \text{pt}$, and*
- (ii) *H -equivariant Lie algebra triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$.*

The proof of Proposition 3.4 is found in Appendix B, but we summarize the construction here. The direction $(ii) \Rightarrow (i)$ is part (b) of Proposition 3.3. In the opposite direction $(i) \Rightarrow (ii)$, let \mathfrak{h} be the Lie algebra of H , and put $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ as a vector space. One finds that \mathfrak{d} carries a unique action Ad of H , extending the adjoint action on $\mathfrak{h} \subseteq \mathfrak{d}$ and such that

$$(10) \quad \iota(\mathfrak{a}(z))\theta_h^R = \text{Ad}_h s(z) - t(z)$$

for all $z \in E_h$. Furthermore, \mathfrak{d} carries a unique Lie bracket such that $\mathfrak{g}, \mathfrak{h}$ are Lie subalgebras and such that the differential of $\text{Ad}: H \rightarrow \text{Aut}(\mathfrak{d})$ gives the adjoint action $\text{ad}: \mathfrak{h} \rightarrow \text{End}(\mathfrak{d})$.

3.2. Dirac Manin triples. If V is a vector space with an element $\beta \in S^2 V$, we denote by $\beta^\sharp: V^* \rightarrow V$ the map $\beta^\sharp(\mu) = \beta(\mu, \cdot)$. A subspace $U \subseteq V$ is called β -coisotropic if $\beta^\sharp(\text{ann}(U)) \subseteq U$.

Definition 3.5. A *Dirac Manin triple* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ is a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ of Lie algebras, together with an element $\beta \in (S^2 \mathfrak{d})^\mathfrak{d}$ such that \mathfrak{g} is β -coisotropic.

If $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is an H -equivariant triple, and β is H -invariant, we call $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ an *H -equivariant Dirac Manin triple*.

If H is simply connected, then the H -equivariance conditions are automatic. If β is non-degenerate and $\mathfrak{g}, \mathfrak{h}$ are both Lagrangian Lie subalgebras, the Dirac Manin triple is an ordinary Manin triple.

We will now associate to any Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ a new Dirac Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$, where γ is non-degenerate and \mathfrak{g} is Lagrangian in \mathfrak{q} . Let \mathfrak{d}_β^* be the Lie algebra, equal to \mathfrak{d}^* as a vector space, with the Lie bracket

$$\langle [\mu_1, \mu_2], \xi \rangle = \langle \mu_2, [\xi, \beta^\sharp(\mu_1)] \rangle, \quad \mu_1, \mu_2 \in \mathfrak{d}_\beta^*, \quad \xi \in \mathfrak{g}.$$

The element β , viewed as a bilinear form on \mathfrak{d}_β^* , is invariant under the bracket. The co-adjoint action of \mathfrak{d} is by derivations of the bracket, hence we may form the semi-direct product

$$\widehat{\mathfrak{d}} = \mathfrak{d} \ltimes \mathfrak{d}_\beta^*.$$

The bilinear form

$$\widehat{\beta}((\xi_1, \mu_1), (\xi_2, \mu_2)) = \beta(\mu_1, \mu_2) + \langle \mu_1, \xi_2 \rangle + \langle \mu_2, \xi_1 \rangle$$

on $\widehat{\mathfrak{d}}$ is invariant and non-degenerate. Note that $\mathfrak{d} \subseteq \mathfrak{d} \ltimes \mathfrak{d}_\beta^*$ is a Lagrangian Lie subalgebra, and \mathfrak{d}_β^* is a Lie algebra ideal. This defines a new Dirac Manin triple $(\mathfrak{d} \ltimes \mathfrak{d}_\beta^*, \mathfrak{d}, \mathfrak{d}_\beta^*)_{\widehat{\beta}}$.

Remark 3.6. As observed by Drinfel'd [9], there is in fact a 1-1 correspondence between (i) Manin pairs $(\widehat{\mathfrak{d}}, \mathfrak{d})$ with a Lie algebra ideal complementary to \mathfrak{d} , and (ii) Lie algebras \mathfrak{d} with invariant elements $\beta \in S^2\mathfrak{d}$. One may interpret this as a classification of \mathcal{CA} -groupoids over $H = \text{pt}$. Here $\widehat{\mathfrak{d}} \equiv \mathfrak{d} \ltimes \mathfrak{d}_\beta^* \rightrightarrows \mathfrak{d}$ is the action Lie groupoid for the translation action of \mathfrak{d}_β^* on \mathfrak{d} via the map $\beta^\sharp: \mathfrak{d}_\beta^* \rightarrow \mathfrak{d}$.

The Lie subalgebra $\mathfrak{c} = \mathfrak{g} \ltimes \mathfrak{d}_\beta^*$ is coisotropic, since it contains the Lagrangian Lie subalgebra $\mathfrak{g} \ltimes \text{ann}(\mathfrak{g})$. Hence \mathfrak{c}^\perp is an ideal in \mathfrak{c} , and the quotient

$$\mathfrak{q} = \mathfrak{c}/\mathfrak{c}^\perp$$

is a Lie algebra with a non-degenerate invariant metric induced from that on $\mathfrak{d} \ltimes \mathfrak{d}_\beta^*$. Let $\gamma \in (S^2\mathfrak{q})^\mathfrak{q}$ be given by the dual metric on \mathfrak{q}^* . The inclusion $\mathfrak{g} \hookrightarrow \mathfrak{d} \ltimes \mathfrak{d}_\beta^*$ descends to an inclusion $\mathfrak{g} \hookrightarrow \mathfrak{q}$ as a Lagrangian Lie subalgebra, thus $(\mathfrak{q}, \mathfrak{g})$ is a Manin pair.

Since \mathfrak{d}_β^* is an ideal complementary to \mathfrak{d} , the same is true of $(\mathfrak{d}_\beta^*)^\perp$. Let $\widehat{f}: \mathfrak{d} \ltimes \mathfrak{d}_\beta^* \rightarrow \mathfrak{d}$ be the projection with kernel $(\mathfrak{d}_\beta^*)^\perp$. Explicitly, $\widehat{f}(\xi, \mu) = \xi + \beta^\sharp(\mu)$. This is a Lie algebra homomorphism, and since $\mathfrak{c}^\perp \subseteq (\mathfrak{d}_\beta^*)^\perp$, it descends to a Lie algebra homomorphism

$$f: \mathfrak{q} \rightarrow \mathfrak{d},$$

with the important properties $f(\xi) = \xi$ for $\xi \in \mathfrak{g}$ and $\beta^\sharp = f \circ f^*$.

Finally, $\mathfrak{r} = f^{-1}(\mathfrak{h})$ is a Lie algebra complement to \mathfrak{g} . We have thus defined a Dirac Manin triple

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma,$$

where γ is non-degenerate and \mathfrak{g} is Lagrangian. We denote by $p_\mathfrak{r} \in \text{End}(\mathfrak{q})$ the projection to \mathfrak{r} along \mathfrak{g} and by $p_\mathfrak{h} \in \text{End}(\mathfrak{d})$ the projection to \mathfrak{h} along \mathfrak{g} ; thus $f \circ p_\mathfrak{r} = p_\mathfrak{h} \circ f$.

Examples 3.7. We describe the triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ associated to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ in some extreme cases.

(i) If $\beta = 0$ one obtains (independent of \mathfrak{h})

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{g} \ltimes \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)_\gamma$$

with γ the bilinear form given by the pairing. The map $f: \mathfrak{q} \rightarrow \mathfrak{d}$ is projection to $\mathfrak{g} \subseteq \mathfrak{d}$.

(ii) If β is non-degenerate, defining a non-degenerate metric on \mathfrak{d} , one finds

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{d} \oplus \overline{\mathfrak{g}/\mathfrak{g}^\perp}, \mathfrak{g}_\Delta, \mathfrak{h} \oplus 0)_\gamma$$

where $\mathfrak{g}/\mathfrak{g}^\perp$ is the quotient Lie algebra with metric induced from that on \mathfrak{d} , and $\overline{\mathfrak{g}/\mathfrak{g}^\perp}$ is the same Lie algebra with the opposite metric. \mathfrak{g}_Δ is embedded ‘diagonally’ as $\xi \mapsto (\xi, [\xi])$ (where $[\xi]$ is the image in $\mathfrak{g}/\mathfrak{g}^\perp$), and the homomorphism f is projection to the first summand.

(iii) In particular, if β is non-degenerate and \mathfrak{g} is *Lagrangian*, we obtain $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$, with f the identity map.

3.3. From Dirac Manin triples to Dirac Lie group structures. Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ be an H -equivariant Dirac Manin triple, and let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ and $f: \mathfrak{q} \rightarrow \mathfrak{g}$ be as in Section 3.2. We will obtain a Dirac Lie group structure (\mathbb{A}, E) on H by reduction from the direct product of the multiplicative Manin pairs

$$(\mathbb{T}H, TH) \times (\overline{\mathfrak{q}} \oplus \mathfrak{q}, \mathfrak{g} \oplus \mathfrak{g}),$$

where $\mathbb{T}H \rightrightarrows \mathfrak{h}^*$ is the standard \mathcal{CA} -groupoid structure, and $\overline{\mathfrak{q}} \oplus \mathfrak{q} \rightrightarrows \mathfrak{q}$ is the pair groupoid.

Proposition 3.8. *The subset $C \subseteq \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$ given as*

$$(11) \quad C = \{(v + \alpha, \zeta, \zeta') \mid \text{Ad}_h f(\zeta') - f(\zeta) = \iota(v)\theta_h^R\}$$

(where $h \in H$ is the base point of $v + \alpha \in \mathbb{T}H$) is a coisotropic, involutive \mathcal{VB} -subgroupoid, with $\mathfrak{a}(C^\perp) = 0$. The reduction of $(\mathbb{T}H, TH) \times (\overline{\mathfrak{q}} \oplus \mathfrak{q}, \mathfrak{g} \oplus \mathfrak{g})$ relative to C is a Dirac Lie group structure (\mathbb{A}, E) .

Proof. An argument similar to that given in the proof of Proposition 3.3 shows that C is a \mathcal{VB} -subgroupoid of rank $\dim \mathfrak{q} + \dim \mathfrak{d}$. Furthermore, C is the pre-image of the \mathcal{LA} -subgroupoid V from Proposition 3.3 under the \mathcal{VB} -groupoid homomorphism

$$(12) \quad \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q}) \rightarrow TH \times (\mathfrak{d} \oplus \mathfrak{d}), \quad (v + \alpha, \zeta, \zeta') \mapsto (v, f(\zeta), f(\zeta')).$$

Since (12) preserves brackets, and since V is a Lie subalgebroid, it follows that C is involutive. The orthogonal bundle C^\perp has rank equal to $\dim \mathfrak{d}$, and is spanned by the sections

$$\psi(\mu) = \left(-\langle \mu, \theta^R \rangle, f^*(\mu), f^*(\text{Ad}_{h^{-1}} \mu) \right), \quad \mu \in \mathfrak{d}^*.$$

Indeed, the pairing with $(v + \alpha, \zeta, \zeta') \in \Gamma(C)$ is $\langle \mu, -\iota(v)\theta_h^R - f(\zeta) + \text{Ad}_h f(\zeta') \rangle = 0$ as required. The property $C^\perp \subseteq C$ follows by checking the definition of C on the sections $\psi(\mu)$,

$$\text{Ad}_h (f(f^*(\text{Ad}_{h^{-1}} \mu))) - f(f^*(\mu)) = 0,$$

using the H -equivariance of $f \circ f^* = \beta^\sharp$. The object space of $\mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$ is $\mathfrak{h}^* \times \mathfrak{q}$, embedded as the space of units $T_e^* H \times \mathfrak{q}_\Delta$. This is contained in C , hence $C^{(0)} = \mathfrak{h}^* \times \mathfrak{q}$. On the other hand, $(C^\perp)^{(0)} \cong \mathfrak{d}^*$, embedded in $C^{(0)}$ by the map $\mathfrak{d}^* \rightarrow \mathfrak{h}^* \times \mathfrak{q}$, $\mu \mapsto (p_h^*(\mu), f^*(\mu))$. We next show that $TH \times (\mathfrak{g} \oplus \mathfrak{g})$ is transverse to C , or equivalently that $TH \times (\mathfrak{g} \oplus \mathfrak{g}) \cap C^\perp$ is trivial. Indeed, vanishing of the T^*H -component of $\psi(\mu)$ is equivalent to $\mu \in \text{ann}(\mathfrak{h})$, but then the last two components are contained in $f^*(\text{ann}(\mathfrak{h})) = \mathfrak{r}^\perp$. Coisotropic reduction by C (cf. Proposition 2.15) gives the multiplicative Manin pair $(\mathbb{A}, E) = ((\mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q}))_C, (TH \times (\mathfrak{g} \oplus$

$\mathfrak{g}))_C$). We have $\mathbb{A}^{(0)} = C^{(0)}/(C^\perp)^{(0)} = (\mathfrak{h}^* \times \mathfrak{q})/\mathfrak{d}^* \cong \mathfrak{g}$ (the last identification is obtained by taking $0 \times \mathfrak{g} \hookrightarrow \mathfrak{h}^* \times \mathfrak{q}$ as a complement to \mathfrak{d}^*), and also $E^{(0)} = \mathfrak{g}$ since

$$(TH \times (\mathfrak{g} \oplus \mathfrak{g}))^{(0)} = \{0\} \times \mathfrak{g} \subseteq \mathfrak{h}^* \times \mathfrak{q}.$$

Since $\mathbb{A}^{(0)} = E^{(0)}$, it follows that (\mathbb{A}, E) is a Dirac Lie group structure on H . \square

The construction of \mathbb{A} by coisotropic reduction defines a \mathcal{CA} -groupoid morphism

$$(13) \quad S: \mathbb{A} \dashrightarrow \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q}).$$

3.4. From Dirac Lie group structures to Dirac Manin triples. In this Section we will show that any Dirac Lie group structure (\mathbb{A}, E) on H arises by the reduction procedure from the last section, from a unique H -equivariant Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$.

3.4.1. Definition of $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. As remarked in Section 3.1, the Dirac structure E is a vacant \mathcal{LA} -groupoid over H . Hence it corresponds to a unique H -equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{q} := \mathbb{A}_e$, and let $f: \mathfrak{q} \rightarrow \mathfrak{d}$ be the linear map given as the sum of the target and anchor map at the group unit $e \in H$,

$$(14) \quad f(\zeta) = t_e(\zeta) + \mathfrak{a}_e(\zeta), \quad \zeta \in \mathfrak{q}.$$

Let $\gamma \in S^2\mathfrak{q}$ be dual to the given metric on \mathbb{A}_e . Write $\mathfrak{q} = \mathfrak{g} \oplus \mathfrak{r}$, with \mathfrak{r} be the kernel of $t_e: \mathbb{A}_e \rightarrow \mathfrak{g}$, and \mathfrak{g} embedded as E_e . Thus $f(\tau) = \mathfrak{a}_e(\tau)$ for $\tau \in \mathfrak{r}$ and $f(\xi) = \xi$ for $\xi \in \mathfrak{g}$.

Define $\beta \in S^2\mathfrak{d}$ by

$$\beta^\sharp = f \circ f^*: \mathfrak{d}^* \rightarrow \mathfrak{d}.$$

This defines $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$, but we will need to show that β is H -invariant and that this triple gives (\mathbb{A}, E) . We will also show that $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ is the triple associated to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. (Among other things, we will have to show that \mathfrak{q} is a Lie algebra and that f is a Lie algebra homomorphism.) As before, we denote by $p_\tau \in \text{End}(\mathfrak{q})$ the projection to \mathfrak{r} along \mathfrak{g} and by $p_\mathfrak{h} \in \text{End}(\mathfrak{d})$ the projection to \mathfrak{h} along \mathfrak{g} . Thus $t_e = 1 - p_\tau$ and $p_\mathfrak{h} \circ f = f \circ p_\tau$.

3.4.2. Trivialization of \mathbb{A} . Since $t, s: E \rightarrow \mathfrak{g}$ are fiberwise isomorphisms, we have $\mathbb{A} = E \oplus \ker(t) = E \oplus \ker(s)$ as vector bundles. Let

$$j: \mathbb{A} \rightarrow E$$

be the projection along $\ker(t)$. The trivialization $E = H \times \mathfrak{g}$ given by the source map $s: E \rightarrow \mathfrak{g}$ extends to a trivialization $\mathbb{A} = H \times \mathfrak{q}$, as follows.

Proposition 3.9. (a) *The map*

$$\mathbb{A} \rightarrow \mathfrak{q}, \quad x \mapsto j(x)^{-1} \circ x$$

defines a trivialization, $\mathbb{A} \cong H \times \mathfrak{q}$, compatible with the metric.

- (b) *The constant sections of $\mathbb{A} \cong H \times \mathfrak{q}$ form a Lie algebra under Courant bracket. Thus \mathfrak{q} inherits a Lie algebra structure.*
- (c) *The subspace \mathfrak{g} is a Lie subalgebra of \mathfrak{q} , and the trivialization of \mathbb{A} restricts to the given trivialization $E \cong H \times \mathfrak{g}$.*
- (d) *The subspace \mathfrak{r} is a Lie subalgebra of \mathfrak{q} , and the trivialization of \mathbb{A} restricts to a trivialization $\ker(t) \cong H \times \mathfrak{r}$.*

- (e) *Restriction of the anchor map to constant sections defines an action $\mathfrak{q} \rightarrow \mathfrak{X}(H)$ with coisotropic stabilizers, so that \mathbb{A} is the corresponding action Courant algebroid (cf. Equation (2)).*

Proof. For each $h \in H$, the map $\mathbb{A}_h \rightarrow \mathfrak{q}$, $x \mapsto j(x)^{-1} \circ x$ has inverse $\mathfrak{q} \rightarrow \mathbb{A}_h$, $\zeta \mapsto y \circ \zeta$, where $y \in E_h$ is the unique element such that $s(y) = t(\zeta)$. It is clear that the resulting trivialization extends that of E . The trivialization is compatible with the metric, since $\langle j(x)^{-1} \circ x, j(x)^{-1} \circ x \rangle = \langle j(x)^{-1}, j(x)^{-1} \rangle + \langle x, x \rangle = \langle x, x \rangle$.

By definition, a section $\sigma \in \Gamma(\mathbb{A})$ is ‘constant’ relative to the trivialization of \mathbb{A} if and only if $\sigma_{h_1 h_2} \circ \sigma_{h_2}^{-1} \in E$, for all h_1, h_2 . This can be rephrased in terms of morphisms: Let $P_E: \mathbb{A} \times \mathbb{A} \dashrightarrow \mathbb{A}$ be the Courant morphism, with underlying map $H \times H \rightarrow H$ projection to the second factor, where $(x_1, x_2) \sim_{P_E} x$ if and only if $x_1 \in E$ and $x = x_2$. Thus $P_E \subseteq \mathbb{A} \times \overline{\mathbb{A}} \times \overline{\mathbb{A}}$ is obtained from $\mathbb{A}_\Delta \times E$ by interchanging the last two components. We note that $\sigma \in \Gamma(\mathbb{A})$ is constant if and only if there is a section $\hat{\sigma} \in \Gamma(\mathbb{A} \times \mathbb{A})$ such that

$$\hat{\sigma} \sim_{\text{Mult}_{\mathbb{A}}} \sigma, \quad \hat{\sigma} \sim_{P_E} \sigma.$$

Note that $\hat{\sigma}$ is uniquely determined by the constant section σ : Its value at h_1, h_2 is $\hat{\sigma}_{h_1, h_2} = (\sigma_{h_1 h_2} \circ \sigma_{h_2}^{-1}, \sigma_{h_2}) \in E_{h_1} \times \mathbb{A}_{h_2}$. Given another constant section σ' , we have

$$[\hat{\sigma}, \hat{\sigma}'] \sim_{\text{Mult}_{\mathbb{A}}} [\sigma, \sigma'], \quad [\hat{\sigma}, \hat{\sigma}'] \sim_{P_E} [\sigma, \sigma'],$$

since Courant morphism preserve Courant brackets. Hence $[\sigma, \sigma']$ is constant. It follows that the space of constant sections is closed under Courant bracket. Furthermore, if σ, σ' are constant, then $[\sigma, \sigma'] + [\sigma', \sigma] = \mathbf{a}^* \mathbf{d} \langle \sigma, \sigma' \rangle = 0$ since $\langle \sigma, \sigma' \rangle$ is constant. Hence the resulting bracket on \mathfrak{q} is skew-symmetric, and hence is a Lie bracket.

It is obvious that the trivialization of \mathbb{A} restricts to the given trivialization of E . Since E is involutive, the constant sections with values in E form a Lie subalgebra, thus \mathfrak{g} is a Lie subalgebra of \mathfrak{q} . On the other hand, $t(x) = 0 \Leftrightarrow t(j(x)) = 0 \Leftrightarrow s(j(x)) = 0 \Leftrightarrow t(j(x)^{-1} \circ x) = 0$, shows that the trivialization takes $\ker(t)$ to \mathfrak{r} . The \mathfrak{r} -valued constant sections σ are exactly those for which $\hat{\sigma} = 0 \times \sigma$. Since this property is preserved under Courant bracket, it follows that \mathfrak{r} is a Lie subalgebra of \mathfrak{q} .

Since the anchor map takes Courant brackets to Lie brackets, we obtain a \mathfrak{q} -action on H . By construction, the Courant bracket on \mathbb{A} extends the Lie bracket on constant sections, and the anchor map extends the action map. As shown in [19] (cf. also Section 1.1), this implies that the action of \mathfrak{q} has coisotropic stabilizers. \square

The first part of the Proposition may be phrased as the statement that the trivializing map $\mathbb{A} \rightarrow \mathfrak{q}$ defines a morphism of Manin pairs

$$(15) \quad T: (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g}),$$

where $x \sim_T \zeta$ if and only if $\zeta = j(x)^{-1} \circ x$.

3.4.3. Construction of the coisotropic subgroupoid $C \subseteq \mathbb{T}H \times (\overline{\mathfrak{q}} \oplus \mathfrak{q})$. In the following discussion, whenever we write a composition of groupoid elements we take it to be implicit that the elements are composable.

Proposition 3.10. *Let (\mathbb{A}, E) be a Dirac Lie group structure on H , and define a Lie algebra structure on $\mathfrak{q} = \mathbb{A}_e$ as above. Then the subset $C \subseteq \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q})$ given as*

$$(16) \quad C = \{(v + \alpha, \zeta, \zeta') \mid \exists x \in \mathbb{A}: v = \mathbf{a}(x), \zeta \circ x \circ \zeta'^{-1} \in E\}$$

is an involutive co-isotropic \mathcal{VB} -subgroupoid satisfying $\mathbf{a}(C^\perp) = 0$. There is a canonical isomorphism of \mathcal{CA} -groupoids $\mathbb{A} \rightarrow C/C^\perp$, taking E to $(\mathbb{T}H \times (\mathfrak{g} \oplus \mathfrak{g})) \cap C / (\mathbb{T}H \times (\mathfrak{g} \oplus \mathfrak{g})) \cap C^\perp$.

Proof. Recall the definition of the division morphism $D: \bar{\mathbb{A}} \times \mathbb{A} \dashrightarrow \mathbb{A}$ from the proof of Proposition 2.3 where $(x_1, x_2) \sim_D x$ if and only if $x_1^{-1} \circ x_2 = x$. Together with the trivialization $T: \mathbb{A} \dashrightarrow \mathfrak{q}$, we obtain a morphism $K = (T \times T) \circ D^\top: \mathbb{A} \dashrightarrow \bar{\mathfrak{q}} \oplus \bar{\mathfrak{q}}$. Under this morphism, $x \sim_K (\zeta_1, \zeta_2)$ if and only if $\zeta_1 \circ x \circ \zeta_2^{-1} \in E$.

Let $R: \mathbb{A} \dashrightarrow \mathbb{T}H \times \mathbb{A}$ be the morphism, with underlying map the diagonal inclusion, defined by the property that $x \sim_R (v + \alpha, y)$ if and only if $v = \mathbf{a}(x)$ and $y - x = \mathbf{a}^*(y)$. Thus $R \subset \mathbb{T}H \times \mathbb{A} \times \bar{\mathbb{A}}$ is obtained from the diagonal morphism cf. Corollary 2.14) by permutation of the components and a sign change of the metric. The composition of R with $\mathbb{T}H_\Delta \times K: \mathbb{T}H \times \mathbb{A} \dashrightarrow \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q})$ is clean, and defines a morphism

$$Q = (\mathbb{T}H_\Delta \times K) \circ R: \mathbb{A} \dashrightarrow \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q})$$

with underlying map $H \rightarrow H$ the identity map. Explicitly,

$$(17) \quad y \sim_Q (v + \alpha, \zeta_1, \zeta_2) \Leftrightarrow \exists x \in \mathbb{A}: \zeta_1 \circ x \circ \zeta_2^{-1} \in E, \quad v = \mathbf{a}(x), \quad y - x = \mathbf{a}^*(\alpha).$$

Since R , K , $\mathbb{T}H_\Delta$ are all \mathcal{CA} -groupoid morphism, the same is true of Q .

We claim that $\ker(Q) = 0$. Indeed, suppose $y \sim_Q (0, 0, 0)$. The condition $x - y = \mathbf{a}^*(\alpha)$ with $\alpha = 0$ gives $x = y$, and the condition $\zeta_1 \circ x \circ \zeta_2^{-1} \in E$ with $\zeta_i = 0$ implies $x = 0$, as claimed. On the other hand, $\text{ran}(Q) = C$. By Lemma 3.11 below, there is an isomorphism of Courant algebroids $\mathbb{A} \rightarrow C/C^\perp$.

Finally, we show that $E = (\mathbb{T}H \times (\mathfrak{g} \oplus \mathfrak{g})) \circ Q$. Suppose (17) with $\alpha = 0$ and $\zeta_1, \zeta_2 \in \mathfrak{g}$. Then $x = y$, and $\zeta_1 \circ y \circ \zeta_2^{-1} \in E$. Since $\zeta_1, \zeta_2 \in \mathfrak{g} = E^{(0)}$ it follows that $y \in E$. Therefore $(\mathbb{T}H \times (\mathfrak{g} \oplus \mathfrak{g})) \circ Q \subseteq E$, and the conclusion follows, since both sides are Lagrangian. \square

Lemma 3.11. *Let $R: \mathbb{A} \dashrightarrow \mathbb{A}'$ be a Courant morphism, with underlying map $\Phi: M \rightarrow M'$ a diffeomorphism. If $\ker(R) = 0$, then $C = \text{ran}(R)$ is co-isotropic, with $\mathbf{a}(C^\perp) = 0$, and $\mathbb{A} \cong \mathbb{A}'_C$ as Courant algebroids. If \mathbb{A}, \mathbb{A}' are \mathcal{CA} -groupoids, and $R: \mathbb{A} \dashrightarrow \mathbb{A}'$ is a \mathcal{CA} -groupoid morphism, then $\mathbb{A} \cong \mathbb{A}'_C$ is an isomorphism of \mathcal{CA} -groupoids.*

Proof. The inclusion $R \subseteq (\mathbb{A}' \times C)|_{\text{gr}(\Phi)}$ shows $(0 \times C^\perp)|_{\text{gr}(\Phi)} \subseteq R^\perp = R$, hence $C^\perp \subseteq C$ so that C is co-isotropic. Furthermore, since $\mathbf{a}(0, y') = (0, \mathbf{a}(y'))$ for $y' \in C^\perp$ is tangent to $\text{gr}(\Phi)$, we see $\mathbf{a}(y') = 0$, hence $\mathbf{a}(C^\perp) = 0$. Let $P: \mathbb{A}' \dashrightarrow \mathbb{A}'_C$ be the Courant morphism defined by the reduction. Thus $y' \sim_P y''$ if and only if $y' \in C$, with y'' its image under the quotient map. We will show that $P \circ R: \mathbb{A} \dashrightarrow \mathbb{A}'_C$ is an isomorphism. Indeed, let $x \in \ker(P \circ R)$. Then $x \sim_R x'$, $x' \sim_P 0$ for some $x' \in \mathbb{A}'$. By definition of P , we have $x' \in C^\perp$. Since $(0 \times C^\perp)|_{\text{gr}(\Phi)} \subseteq R$, $x \sim_R x'$ implies $x \sim_R 0$, hence $x = 0$. The property $\ker(P \circ R) = 0$, $\text{ran}(P \circ R) = \mathbb{A}'$ means that $P \circ R$ defines an isomorphism $\mathbb{A} \cong \mathbb{A}'_C$. If $R: \mathbb{A} \dashrightarrow \mathbb{A}'$ is a morphism of \mathcal{CA} -groupoids, then so is P and hence $P \circ R$. \square

The co-isotropic subbundle C has an alternative description, similar to Proposition 3.8.

Proposition 3.12. *The co-isotropic subbundle $C \subseteq \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q})$ from Proposition 3.10 may be written,*

$$C = \{(v + \alpha, \zeta_1, \zeta_2) \mid \iota(v)\theta_h^R = \text{Ad}_h f(\zeta_2) - f(\zeta_1)\}.$$

Proof. Given $(v + \alpha, \zeta_1, \zeta_2) \in \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q})$, let $z \in E_h$ be the unique element with $s(z) = t(\zeta_2)$. By Equation (10), we have $\text{Ad}_h t(\zeta_2) = \text{Ad}_h s(z) = \iota(\mathfrak{a}(z))\theta_h^R + t(z)$. Together with Equation (14) we obtain

$$\begin{aligned} \text{Ad}_h f(\zeta_2) - f(\zeta_1) &= \text{Ad}_h(t(\zeta_2) + \mathfrak{a}(\zeta_2)) - (t(\zeta_1) + \mathfrak{a}(\zeta_1)) \\ &= \iota(\mathfrak{a}(z))\theta_h^R + \text{Ad}_h \mathfrak{a}(\zeta_2) - \mathfrak{a}(\zeta_1) + t(z) - t(\zeta_1). \end{aligned}$$

The first three terms lie in \mathfrak{h} , the last two in \mathfrak{g} . Hence the property $\iota(v)\theta_h^R = \text{Ad}_h f(\zeta_2) - f(\zeta_1)$ is equivalent to the two conditions

$$(18) \quad \iota(v)\theta_h^R = \iota(\mathfrak{a}(z))\theta_h^R + \text{Ad}_h \mathfrak{a}(\zeta_2) - \mathfrak{a}(\zeta_1), \quad t(\zeta_1) + t(z)$$

Equation (18) is equivalent to the condition that $x := \zeta_1^{-1} \circ z \circ \zeta_2$ is defined and $v = \mathfrak{a}(x)$. \square

Remark 3.13. Define a bundle map $H \times \mathfrak{q} \rightarrow C$, $(h, \zeta) \mapsto (v, \xi, \zeta)$ where $\iota(v)\theta_h^R = p(\text{Ad}_h f(\zeta))$ and $\xi = (1-p)\text{Ad}_h f(\zeta)$. The sub-bundle given as its image is invariant under left groupoid multiplication by elements of $\mathbb{T}H \times (\mathfrak{g} \oplus \mathfrak{g})$, and is a complement to C^\perp . Hence, its composition with the quotient map to $\mathbb{A} = C/C^\perp$ is the trivialization $H \times \mathfrak{q} \cong \mathbb{A}$ from Proposition 3.9.

3.4.4. *Relation between $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ and $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$.* We still have to show that β is H -invariant, and that $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ is the Dirac Manin triple associated to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ by the construction from Section 3.2.

Proposition 3.14. (a) *The map $f: \mathfrak{q} \rightarrow \mathfrak{d}$ is a Lie algebra homomorphism.*

(b) *The element $\beta \in S^2\mathfrak{d}$ defined by $\beta^\sharp = f \circ f^*$ is \mathfrak{d} -invariant as well as Ad_h -invariant.*

(c) *\mathfrak{g} is β -coisotropic.*

(d) *The Lie subalgebra $\mathfrak{c} = \mathfrak{g} \ltimes \mathfrak{d}_\beta^* \subseteq \mathfrak{d} \ltimes \mathfrak{d}_\beta^*$ is coisotropic, and the map*

$$\mathfrak{c} \rightarrow \mathfrak{q}, \quad (\xi, \mu) \mapsto \xi + f^*(\mu)$$

descends to an isometric Lie algebra isomorphism $\mathfrak{c}/\mathfrak{c}^\perp \rightarrow \mathfrak{q}$.

Proof. (a) Since f restricts to a Lie algebra homomorphism $\mathfrak{r} \rightarrow \mathfrak{h}$, and is given by the identity map on \mathfrak{g} , we need only check that $f([\tau, \xi]) = [f(\tau), \xi]$ for $\xi \in \mathfrak{g}$, $\tau \in \mathfrak{r}$. Define sections of $C \subseteq \mathbb{T}H \times (\mathfrak{q} \oplus \mathfrak{q})$ by

$$(f(\tau))^L, 0, \tau, \quad (p_{\mathfrak{h}}(\text{Ad}_h \xi))^R, (1 - p_{\mathfrak{h}})\text{Ad}_h \xi, \xi.$$

Since C is involutive, their Courant bracket

$$(p_{\mathfrak{h}}(\text{Ad}_h [f(\tau), \xi]))^R, (1 - p_{\mathfrak{h}})\text{Ad}_h [f(\tau), \xi], [\tau, \xi]$$

is again a section of C . Thus

$$\begin{aligned} p_{\mathfrak{h}}(\text{Ad}_h [f(\tau), \xi]) &= \text{Ad}_h f([\tau, \xi]) - (1 - p_{\mathfrak{h}})\text{Ad}_h [f(\tau), \xi] \\ &= \text{Ad}_h f([\tau, \xi]) - \text{Ad}_h [f(\tau), \xi] + p_{\mathfrak{h}}(\text{Ad}_h [f(\tau), \xi]), \end{aligned}$$

giving $f([\tau, \xi]) = [f(\tau), \xi]$ as desired.

- (b) By the same argument as in the proof of Proposition 3.8, the fiber of C^\perp at $h \in H$ is spanned by the sections $\psi(\mu)$, $\mu \in \mathfrak{d}^*$. The property $C^\perp \subseteq C$ gives $0 = \text{Ad}_h(f(f^*(\mu)) - f(f^*(\mu)))$ as desired. This shows that β is invariant under the adjoint action of H . In particular it is \mathfrak{h} -invariant. Since f is a Lie algebra homomorphism, it is also equivariant under the adjoint action of \mathfrak{g} . Thus $\beta = f \circ f^*$ is \mathfrak{g} -invariant as well.
- (c) The dual map $f^*: \mathfrak{d}^* \rightarrow \mathfrak{q}$ takes $\text{ann}(\mathfrak{g}) \subseteq \mathfrak{d}^*$ to $\mathfrak{g}^\perp \subseteq \mathfrak{q}$. Hence, for $\mu \in \text{ann}(\mathfrak{g})$, $\beta(\mu, \mu) = \langle f^*(\mu), f^*(\mu) \rangle = 0$.
- (d) The map $\mathfrak{c} \rightarrow \mathfrak{q}$, $(\xi, \mu) \mapsto \xi + f^*(\mu)$ is surjective, since its image contains \mathfrak{g} as well as the complement $f^*(\text{ann}(\mathfrak{h})) = \mathfrak{r}^\perp$. The map clearly preserves the bilinear forms, hence its kernel must be \mathfrak{c}^\perp . Using the identity

$$[f^*(\mu_1), f^*(\mu_2)] = f^*([\beta^\sharp(\mu_1), \mu_2]),$$

(which is verified by pairing both sides with $\zeta \in \mathfrak{q}$), one finds that it is a Lie algebra homomorphism. □

It follows that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ is an H -equivariant Dirac Manin triple, and that $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ and $f: \mathfrak{q} \rightarrow \mathfrak{g}$ result from the construction of Section 3.2. Propositions 3.10 and 3.12 define a canonical isomorphism between (\mathbb{A}, E) and the Dirac Lie group constructed in Section 3.3.

4. MORPHISMS

In this section, we will show that the correspondence between Dirac Lie group structures and H -equivariant Dirac Manin pairs respects morphisms, thus completing the proof of Theorem 0.1.

4.1. Morphisms of Dirac Manin triples. The Dirac Manin triples form a category relative to the following notion of morphism.

Definition 4.1. A morphism of Dirac Manin triples

$$\mathfrak{k}: (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$$

is a $\beta_1 - \beta_0$ -coisotropic Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$ such that

$$\mathfrak{g}_1 = \mathfrak{k} \circ \mathfrak{g}_0, \quad \mathfrak{h}_0 = \mathfrak{h}_1 \circ \mathfrak{k}.$$

See Appendix A for compositions of linear relations. The property $\mathfrak{h}_0 = \mathfrak{h}_1 \circ \mathfrak{k}$ implies $\ker(\mathfrak{k}) \subseteq \mathfrak{h}_0$, hence $\mathfrak{g}_0 \cap \ker(\mathfrak{k}) = 0$. Hence there is a linear map $\psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$, taking $\xi_1 \in \mathfrak{g}_1$ to the unique element $\xi_0 = \psi(\xi_1) \in \mathfrak{g}_0$ with $\xi_0 \sim_{\mathfrak{k}} \xi_1$. By a similar argument, there is a linear map $\phi: \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$, taking $\nu_0 \in \mathfrak{h}_0$ to the unique element $\nu_1 = \phi(\nu_0) \in \mathfrak{h}_1$ with $\nu_0 \sim_{\mathfrak{k}} \nu_1$.

Lemma 4.2. \mathfrak{k} is the direct sum of the graphs of ψ, ϕ . In particular

$$\dim \mathfrak{k} = \dim \mathfrak{g}_1 + \dim(\mathfrak{d}_0/\mathfrak{g}_0).$$

Proof. Suppose $\xi_0 + \nu_0 \sim_{\mathfrak{k}} \xi_1 + \nu_1$ with $\xi_i \in \mathfrak{g}_i$ and $\nu_i \in \mathfrak{h}_i$. Since $\psi(\xi_1) \sim_{\mathfrak{k}} \xi_1$ and $\nu_0 \sim \phi(\nu_0)$, it follows that $\xi_0 - \psi(\xi_1) \sim_{\mathfrak{k}} \nu_1 - \phi(\nu_0)$. Since $\mathfrak{h}_0 = \mathfrak{h}_1 \circ \mathfrak{k}$, this implies that $\xi_0 - \psi(\xi_1) \in \mathfrak{h}_0$, hence $\xi_0 - \psi(\xi_1) = 0$. Similarly $\nu_1 - \phi(\nu_0) = 0$. □

Remark 4.3. Suppose that the β_i are non-degenerate and that the Lie subalgebras \mathfrak{g}_i are Lagrangian. Then $\mathfrak{k} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$ is $\beta_1 - \beta_0$ -Lagrangian, for dimensional reasons. For all $(\xi_1 + \phi(\nu_0), \psi(\xi_1) + \nu_0) \in \mathfrak{k}$, we find (denoting the metric on \mathfrak{d}_i simply by $\langle \cdot, \cdot \rangle$),

$$\begin{aligned} 0 &= \langle (\xi_1 + \phi(\nu_0), \psi(\xi_1) + \nu_0), (\xi_1 + \phi(\nu_0), \psi(\xi_1) + \nu_0) \rangle \\ &= 2\langle \phi(\nu_0), \xi_1 \rangle + \langle \phi(\nu_0), \phi(\nu_0) \rangle - 2\langle \psi(\xi_1), \nu_0 \rangle - \langle \nu_0, \nu_0 \rangle. \end{aligned}$$

Since this is true for all $\xi_1 \in \mathfrak{g}_1$, this shows that $\psi = \phi^*$, for the identification $\mathfrak{h}_i = \mathfrak{g}_i^*$ given by the pairing, and that ϕ preserves the induced bilinear forms on \mathfrak{h}_i . In particular, for ordinary Manin triples $(\mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i)$, a morphism is given by a pair of Lie algebra homomorphisms $\psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ and $\phi: \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$ such that $\phi = \psi^*$.

Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ be a Dirac Manin triple, and $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ the Dirac Manin triple obtained by the construction from Section 3.2, with the corresponding map $f: \mathfrak{q} \rightarrow \mathfrak{d}$. Then the graph of f defines a morphism of Dirac Manin triples

$$\text{gr}(f): (\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma \dashrightarrow (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta.$$

The construction of $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ is functorial:

Proposition 4.4. *Let $\mathfrak{k}: (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$ be a morphism of Dirac Manin triples, given as the direct sum of the graphs of $\phi: \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$ and $\psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$. Let $(\mathfrak{q}_i, \mathfrak{g}_i, \mathfrak{r}_i)_{\gamma_i}$ be the Dirac Manin triples associated to $(\mathfrak{d}_i, \mathfrak{g}_i, \mathfrak{h}_i)_{\beta_i}$ as in Section 3.2, with the corresponding maps $f_i: \mathfrak{q}_i \rightarrow \mathfrak{d}_i$. Let $\mathfrak{l} \subseteq \mathfrak{q}_1 \times \mathfrak{q}_0$ be the direct sum of the graphs of ψ and $\kappa = \psi^*: \mathfrak{r}_0 = \mathfrak{g}_0^* \rightarrow \mathfrak{r}_1 = \mathfrak{g}_1^*$. Then \mathfrak{l} defines a morphism of Dirac Manin triples*

$$\mathfrak{l}: (\mathfrak{q}_0, \mathfrak{g}_0, \mathfrak{r}_0)_{\gamma_0} \dashrightarrow (\mathfrak{q}_1, \mathfrak{g}_1, \mathfrak{r}_1)_{\gamma_1},$$

with $f(\mathfrak{l}) \subseteq \mathfrak{k}$ where $f = f_1 \times f_0$. As a consequence,

$$\text{gr}(f_1) \circ \mathfrak{l} = \mathfrak{k} \circ \text{gr}(f_0).$$

Proof. We will begin with an alternative construction of \mathfrak{l} by reduction. Write $\mathfrak{d} = \mathfrak{d}_1 \times \mathfrak{d}_0$, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_0$, $\mathfrak{h} = \mathfrak{h}_1 \times \mathfrak{h}_0$, and $\beta = \beta_1 - \beta_0$ so that $\mathfrak{k} \subseteq \mathfrak{d}$ is a β -coisotropic Lie subalgebra. Then

$$\widehat{\mathfrak{k}} = \mathfrak{k} \ltimes \text{ann}(\mathfrak{k}) \subseteq \mathfrak{d} \ltimes \mathfrak{d}_\beta^*$$

is a Lagrangian Lie subalgebra. Since \mathfrak{k} is the direct sum of the graphs of $\psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ and $\phi: \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$, its annihilator $\text{ann}(\mathfrak{k})$ is the direct sum of the graphs of $\phi^*: \text{ann}(\mathfrak{g}_1) \rightarrow \text{ann}(\mathfrak{g}_0)$ and $\psi^*: \text{ann}(\mathfrak{h}_0) \rightarrow \text{ann}(\mathfrak{h}_1)$. Let $\mathfrak{l} \subseteq \mathfrak{q}$ be the Lagrangian Lie subalgebra defined by reduction of $\widehat{\mathfrak{k}}$ with respect to the coisotropic Lie subalgebra $\mathfrak{c} = \mathfrak{g} \ltimes \mathfrak{d}_\beta^*$,

$$\mathfrak{l} = \widehat{\mathfrak{k}} \cap \mathfrak{c} / \widehat{\mathfrak{k}} \cap \mathfrak{c}^\perp \subseteq \mathfrak{q}.$$

Since \mathfrak{k} is coisotropic, we have $\beta^\sharp(\text{ann}(\mathfrak{k})) \subseteq \mathfrak{k}$, and hence $f(\mathfrak{l}) \subseteq \mathfrak{k}$, by definition of f . Equivalently, $\mathfrak{k} \supseteq \text{gr}(f_1) \circ \mathfrak{l} \circ \text{gr}(f_0)^\top$. This implies

$$\mathfrak{k} \circ \text{gr}(f_0) \supseteq \text{gr}(f_1) \circ \mathfrak{l} \circ \text{gr}(f_0)^\top \circ \text{gr}(f_0) \supseteq \text{gr}(f_1) \circ \mathfrak{l}.$$

The inclusion

$$\mathfrak{r}_1 \circ \mathfrak{l} = \mathfrak{h}_1 \circ \text{gr}(f_1) \circ \mathfrak{l} \subseteq \mathfrak{h}_1 \circ \mathfrak{k} \circ \text{gr}(f_0) = \mathfrak{h}_0 \circ \text{gr}(f_0) = \mathfrak{r}_0$$

shows $\mathfrak{r}_1 \circ \mathfrak{l} = \mathfrak{r}_0$, since both sides are Lagrangian. On the other hand, since $\widehat{\mathfrak{k}} \cap \mathfrak{c} \supseteq \mathfrak{k} \cap \mathfrak{g} = \text{gr}(\psi)$, we have $\mathfrak{l} \supseteq \text{gr}(\psi)$, and hence

$$\mathfrak{l} \circ \mathfrak{g}_0 \supseteq \text{gr}(\psi) \circ \mathfrak{g}_0 \supseteq \mathfrak{g}_1.$$

Since both sides are Lagrangian, it follows that $\mathfrak{l} \circ \mathfrak{g}_0 = \mathfrak{g}_1$. This shows that \mathfrak{l} defines a morphism $\mathfrak{l}: (\mathfrak{q}_0, \mathfrak{g}_0, \mathfrak{r}_0)_{\gamma_0} \dashrightarrow (\mathfrak{q}_1, \mathfrak{g}_1, \mathfrak{r}_1)_{\gamma_1}$ of Dirac Manin triples. By Remark 4.3, \mathfrak{l} is the direct sum of the graph of ψ with a Lie algebra homomorphism $\kappa = \psi^*$, recovering the description of \mathfrak{l} given in the proposition.

Finally we show that $\text{gr}(f_1) \circ \mathfrak{l} = \mathfrak{k} \circ \text{gr}(f_0)$, or equivalently

$$(19) \quad \phi \circ f_0 = f_1 \circ \kappa, \quad \psi \circ f_1 = f_0 \circ \psi$$

For the first equation, let $y \in \mathfrak{r}_0$. Then $\kappa(y) + y \in \mathfrak{r} \cap \mathfrak{l}$, and $f_1(\kappa(y)) + f_0(y) \in \mathfrak{h} \cap \mathfrak{k}$. Hence $\phi(f_0(y)) = f_1(\kappa(y))$ by definition of ϕ . Similarly, given $x \in \mathfrak{g}_1$ we have $x + \psi(x) \in \mathfrak{g} \cap \mathfrak{l}$, hence $f_1(x) + f_0(\psi(x)) \in \mathfrak{g} \cap \mathfrak{k}$. Thus $\psi(f_1(x)) = f_0(\psi(x))$. \square

4.2. The Equivalence Theorem for morphisms. A morphism of H_i -equivariant Dirac Manin triples $\mathfrak{k}: (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$ is a morphism of Dirac Manin triples where the map $\phi: \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$ is the differential of a Lie group homomorphism $\Phi: H_0 \rightarrow H_1$, and where \mathfrak{k} is invariant under the adjoint action of $H_0 \cong \text{gr}(\Phi) \subseteq H_1 \times H_0$. Our goal is to show that such morphisms are in 1-1 correspondence with the morphisms of the corresponding Dirac Lie group structures. (Recall that a morphism of Dirac Lie groups is defined to be a morphism of multiplicative Manin pairs.)

Theorem 4.5 (Morphisms). *There is a 1-1 correspondence between morphisms of H_i -equivariant Dirac Manin triples $\mathfrak{k}: (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$ and morphisms of the corresponding Dirac Lie groups, $L: (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$. This correspondence is compatible with the composition of morphisms.*

The proof of this Theorem will be given in the next two subsections. We will use the presentation $\mathbb{A}_i = C_i / C_i^\perp$ with $C_i \subset \mathbb{T}H \times (\overline{\mathfrak{q}}_i \oplus \mathfrak{q}_i)$, as in Section 3.3. For convenience, let us write $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_0$, $C = C_1 \times C_0$, $H = H_1 \times H_0$, $\mathfrak{q} = \overline{\mathfrak{q}}_1 \times \mathfrak{q}_0$, $\mathfrak{d} = \mathfrak{d}_1 \times \mathfrak{d}_0$, $\beta = \beta_1 - \beta_0$, $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_0$ etc.

4.2.1. The direction \Rightarrow . Given $\mathfrak{k}: (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$, let $\mathfrak{l}: (\mathfrak{q}_0, \mathfrak{g}_0, \mathfrak{r}_0)_{\gamma_0} \dashrightarrow (\mathfrak{q}_1, \mathfrak{g}_1, \mathfrak{r}_1)_{\gamma_1}$ be the morphism constructed in Proposition 4.4, and consider the Courant morphism

$$R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l}): \mathbb{T}H_0 \times (\overline{\mathfrak{q}}_0 \oplus \mathfrak{q}_0) \dashrightarrow \mathbb{T}H_1 \times (\overline{\mathfrak{q}}_1 \oplus \mathfrak{q}_1).$$

Let $S_i: \mathbb{A}_i \dashrightarrow \mathbb{T}H_i \times (\overline{\mathfrak{q}}_i \oplus \mathfrak{q}_i)$ be the Courant morphisms defined by the reduction, as in (13).

Lemma 4.6. *The composition $L = S_1^\top \circ (R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l})) \circ S_0$ defines a Dirac Lie group morphism*

$$(20) \quad L: (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1).$$

The trivialization $\mathbb{A} \rightarrow \mathfrak{q}$ from Proposition 3.9 restricts to a trivialization $L \rightarrow \mathfrak{l}$.

Proof. By definition,

$$L = (R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l}))_C \subseteq \mathbb{A} = (\mathbb{T}H \times (\bar{\mathfrak{d}} \oplus \mathfrak{d}))_C.$$

We will show that the trivializing map $\mathbb{A} \rightarrow \mathfrak{q}$ restricts to $L \rightarrow \mathfrak{l}$, defining an isomorphism $L \xrightarrow{\cong} \text{gr}(\Phi) \times \mathfrak{l}$. Since $\mathbb{A}_i \xrightarrow{\cong} H_i \times \mathfrak{q}_i$ restricts to $E_i \xrightarrow{\cong} H_i \times \mathfrak{g}_i$, this then implies that L gives a morphism of Manin pairs $(\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$.

We first verify $L_e = \mathfrak{l}$. By (11), the fiber C_e consists of elements $(f(\zeta') - f(\zeta) + \mu, \zeta, \zeta')$ with $\mu \in \mathfrak{h}^*$ and $\zeta, \zeta' \in \mathfrak{q}$. Given $\zeta \in \mathfrak{l}$, we have $t(\zeta) + \mathfrak{a}(\zeta) = f(\zeta) \in \mathfrak{k}$ by Equation (14). Hence $t(\zeta) \in \mathfrak{k} \cap \mathfrak{g} = \mathfrak{l} \cap \mathfrak{g}$ and $\mathfrak{a}(\zeta) \in \mathfrak{k} \cap \mathfrak{h}$. It follows that

$$(21) \quad (\mathfrak{a}(\zeta), t(\zeta), \zeta) \in C_e \cap (R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l})),$$

mapping to $\zeta \in C_e/C_e^\perp$ (cf. Remark 3.13).

This shows $L_e \supseteq \mathfrak{l}$, and equality follows since both sides are Lagrangian.

Recall next that on $E \subset \mathbb{A}$, the trivialization is given by the source map $s: E \rightarrow \mathfrak{g}$. Suppose $z \in E_h$ satisfies $s(z) \in \mathfrak{l}$. Then $s(z) \in \mathfrak{g} \cap \mathfrak{l} = \mathfrak{g} \cap \mathfrak{k}$. Equation (10) shows that

$$\text{Ad}_h s(z) = t(z) + \iota(\mathfrak{a}(z))\theta_h^R.$$

Since \mathfrak{k} is invariant under the action of $\text{gr}(\Phi) \subseteq H_1 \times H_0$, the left hand side lies in \mathfrak{k} , hence so does the right hand side. Since \mathfrak{k} is the direct sum of its intersections with $\mathfrak{h}, \mathfrak{g}$, this shows $t(z) \in \mathfrak{g} \cap \mathfrak{k} = \mathfrak{g} \cap \mathfrak{l}$ and $\mathfrak{a}(z) \in T \text{gr}(\Phi)$. Hence $(\mathfrak{a}(z), t(z), s(z)) \in C \cap R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l})$ maps to $z \in E$, proving that $z \in E \cap L$.

Finally, given $h \in \text{gr}(\Phi)$ and $\zeta \in \mathfrak{l}$, let $z \in E_h$ be the unique element such that $t(\zeta) = s(z)$. Then $x := z \circ \zeta \in \mathbb{A}_h$ maps to ζ under the trivialization. Since $\zeta \in L_e$ and $z \in L_h$, by the above, it follows that $x \in L_h$. This shows that the image of L_h under $\mathbb{A} \rightarrow \mathfrak{q}$ contains \mathfrak{l} , and equality follows since both sides are Lagrangian. \square

4.2.2. The direction \Leftarrow . Suppose $L: (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$ is a morphism of Dirac Lie groups, with underlying Lie group homomorphism $\Phi: H_0 \rightarrow H_1$. We will show how to construct \mathfrak{k} .

Proposition 4.7. *Suppose that*

$$L: (\mathbb{A}_0, E_0) \dashrightarrow (\mathbb{A}_1, E_1)$$

is a morphism of multiplicative Manin pairs over the morphism $\Phi: H_0 \rightarrow H_1$ of Lie groups. Then there exists a unique H -equivariant morphism

$$\mathfrak{k}: (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$$

of Dirac Manin triples such that L results from the construction of Section 4.2.1.

Proof. We begin by trivializing the bundle L . By definition of a morphism of Manin pairs, any element of $(E_1)_{\Phi(h_0)}$ is $L_{(\Phi(h_0), h_0)}$ -related to a unique element of $(E_0)_{h_0}$. This shows that $L \cap E$ is of constant rank equal to $\text{rank}(E_1)$, and hence is a \mathcal{VB} -subgroupoid. Let $\mathfrak{l} = L_e$ be the fiber at the identity. The space of objects of L is $L^{(0)} = L \cap A^{(0)} = \mathfrak{l} \cap \mathfrak{g}$, and coincides with the space of objects of $L \cap E$. Thus $L \cap E$ is a wide subgroupoid of L , and $L = (L \cap E) \oplus \ker(t|_L)$. It follows that the projection $j: \mathbb{A} \rightarrow E$ along $\ker(t)$ restricts to a projection $j: L \rightarrow L \cap E$, and that the map $\mathbb{A} \rightarrow \mathfrak{q}$, $x \mapsto j(x)^{-1} \circ x$ restricts to a

trivialization $L \rightarrow \mathfrak{l}$. Equivalently, the isomorphism $\mathbb{A} \xrightarrow{\cong} H \times \mathfrak{q}$ restricts to an isomorphism $L \xrightarrow{\cong} \text{gr}(\Phi) \times \mathfrak{l}$, with $L \cap E \xrightarrow{\cong} \text{gr}(\Phi) \times (\mathfrak{l} \cap \mathfrak{g})$ and $L \cap \ker(t) \xrightarrow{\cong} \text{gr}(\Phi) \times (\mathfrak{l} \cap \mathfrak{r})$.

Since L is a Dirac structure along $\text{gr}(\Phi)$, it follows that $\mathfrak{l} \subset \mathfrak{q}$ is a Lie subalgebra. Letting $T_i: (\mathbb{A}_i, E_i) \dashrightarrow (\mathfrak{q}_i, \mathfrak{g}_i)$ be the trivializations described in Proposition 3.9, we have shown that

$$\begin{array}{ccc} (\mathbb{A}_0, E_0) & \xrightarrow{L} & (\mathbb{A}_1, E_1) \\ \downarrow T_0 & & \downarrow T_1 \\ (\mathfrak{q}_0, \mathfrak{g}_0) & \xrightarrow{\mathfrak{l}} & (\mathfrak{q}_1, \mathfrak{g}_1) \end{array}$$

is a commutative diagram of morphisms of Manin pairs. From $\mathfrak{l} = \mathfrak{l} \cap \mathfrak{g} \oplus \mathfrak{l} \cap \mathfrak{r}$, and the fact that $\mathfrak{l} \cap \mathfrak{g}$ is the graph of a linear map $\psi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$, it follows that $\mathfrak{l} \cap \mathfrak{r}$ is the graph of a linear map $\kappa = \psi^*: \mathfrak{r}_0 \rightarrow \mathfrak{r}_1$. Hence \mathfrak{l} defines a morphism of Dirac Manin triples $\mathfrak{l}: (\mathfrak{q}_0, \mathfrak{g}_0, \mathfrak{r}_0)_{\gamma_0} \dashrightarrow (\mathfrak{q}_1, \mathfrak{g}_1, \mathfrak{r}_1)_{\gamma_1}$.

We have to verify that L is recovered from \mathfrak{l} by the construction from the last subsection. As in the proof of Proposition 3.10, consider the multiplicative Courant relations

$$R_i: \mathbb{A}_i \dashrightarrow \mathbb{T}H_i \times \mathbb{A}_i, \quad D_i^\top: \mathbb{A}_i \dashrightarrow \overline{\mathbb{A}_i} \times \mathbb{A}_i$$

for $i = 0, 1$, where $\mathbb{A}_i \times \overline{\mathbb{A}_i}$ is taken to be the pair groupoid. Thus

$$v + \alpha \sim_{R_i} (y, x) \Leftrightarrow x - y = \mathfrak{a}^* \alpha, \quad v = \mathfrak{a}(x), \quad x \sim_{D_i^\top} (x_1, x_2) \Leftrightarrow x = x_1^{-1} \circ x_2.$$

We also have the identity morphisms $(\mathbb{T}H_i)_\Delta: \mathbb{T}H_i \dashrightarrow \mathbb{T}H_i$, the standard lift $R_\Phi: \mathbb{T}H_0 \dashrightarrow \mathbb{T}H_1$ of Φ defined by (4), and the Courant morphism $\mathfrak{l} = L_e: \mathfrak{q}_0 \dashrightarrow \mathfrak{q}_1$. These compose to form the following diagram of multiplicative Courant relations,

$$\begin{array}{ccc} \mathbb{A}_0 & \xrightarrow{\quad L \quad} & \mathbb{A}_1 \\ \downarrow R_0 & & \downarrow R_1 \\ \mathbb{T}H_0 \times \mathbb{A}_0 & \xrightarrow{\quad R_\Phi \times L \quad} & \mathbb{T}H_1 \times \mathbb{A}_1 \\ \downarrow (\mathbb{T}H_0)_\Delta \times D_0^\top & & \downarrow (\mathbb{T}H_1)_\Delta \times D_1^\top \\ \mathbb{T}H_0 \times (\overline{\mathbb{A}_0} \times \mathbb{A}_0) & \xrightarrow{\quad R_\Phi \times L \times L \quad} & \mathbb{T}H_1 \times (\overline{\mathbb{A}_1} \times \mathbb{A}_1) \\ \downarrow (\mathbb{T}H_0)_\Delta \times T_0 \times T_0 & & \downarrow (\mathbb{T}H_1)_\Delta \times T_1 \times T_1 \\ \mathbb{T}H_0 \times (\overline{\mathfrak{q}_0} \oplus \mathfrak{q}_0) & \xrightarrow{\quad R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l}) \quad} & \mathbb{T}H_1 \times (\overline{\mathfrak{q}_1} \oplus \mathfrak{q}_1) \end{array}$$

Let $Q_i: \mathbb{A}_i \dashrightarrow \mathbb{T}H_i \times (\overline{\mathfrak{q}_i} \oplus \mathfrak{q}_i)$ be the vertical composition of relations, as in Proposition 3.10. In the diagram above, the bottom square commutes, since the trivialization of \mathbb{A} restricts to the trivialization $L \rightarrow \text{gr}(\Phi) \times \mathfrak{l}$. The middle square commutes since $L: \mathbb{A}_0 \dashrightarrow \mathbb{A}_1$ is a multiplicative Courant morphism, and D_i is the graph of division for the groupoid \mathbb{A}_i . Finally, the top square commutes since both $R_1 \circ L$ and $(R_\Phi \times L) \circ R_0$ are the morphism $\mathbb{A}_0 \dashrightarrow \mathbb{T}H_1 \times \mathbb{A}_1$, where x is related to $(v + \alpha, y)$ if and only if $x \sim_L y + \mathfrak{a}^* \alpha, v = \mathfrak{a}(y)$. We conclude that

$$(R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l})) \circ Q_0 = Q_1 \circ L.$$

Now, as in Proposition 3.10, $\ker Q_1 = 0$. Consequently, $Q_1^\top \circ Q_1 = (\mathbb{A}_1)_\Delta$, the identity relation for \mathbb{A}_1 . Hence

$$(22) \quad Q_1^\top \circ (R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l})) \circ Q_0 = Q_1^\top \circ Q_1 \circ L = L.$$

Let $Q = Q_1 \times Q_0 : \mathbb{A} \dashrightarrow \mathbb{T}H \times (\bar{\mathfrak{q}} \oplus \mathfrak{q})$, and $C = \text{ran}(Q)$. Since $\ker Q = 0$, Lemma 3.11 and (22) imply that

$$L = (R_\Phi \times (\mathfrak{l} \oplus \mathfrak{l}))_C \subseteq \mathbb{A} = (\mathbb{T}H \times (\bar{\mathfrak{d}} \oplus \mathfrak{d}))_C,$$

as claimed.

It remains to construct $\mathfrak{k} : (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$. Recall again that $L \cap E$ is a vacant \mathcal{LA} -groupoid. By Proposition 3.4, it corresponds to a $\text{gr}(\Phi) \cong H_0$ -equivariant Lie algebra triple $(\mathfrak{k}, \mathfrak{g}', \mathfrak{h}')$. Since $L \cap E$ is a subgroupoid of the vacant \mathcal{LA} -groupoid E , we have $\mathfrak{g}' \subseteq \mathfrak{g}$, $\mathfrak{h}' \subseteq \mathfrak{h}$, and $\mathfrak{k} \subseteq \mathfrak{d}$ is a $\text{gr}(\Phi)$ -invariant Lie subalgebra. Here \mathfrak{h}' is the Lie algebra of the base $H' = \text{gr}(\Phi) \subseteq H$, i.e. $\mathfrak{h}' = \text{gr}(\phi)$ with $\phi = d_e \Phi$. On the other hand, we saw that $\mathfrak{g}' = \mathfrak{l} \cap \mathfrak{g}$ is the graph of a Lie algebra homomorphism $\psi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$. Since $L \cap E$ is supported on $\text{gr}(\Phi)$ and $t(L \cap E) = (L \cap E)^{(0)} = \mathfrak{l} \cap \mathfrak{g}$, it follows from (14) that $f(\mathfrak{l}) \subseteq \mathfrak{k}$. Then the fact that $\mathfrak{k} \subseteq \mathfrak{d}$ is co-isotropic is a special case of Lemma 4.8 below.

We conclude that $\mathfrak{k} \subseteq \mathfrak{d}$ defines an H -equivariant morphism of Dirac Manin triples $\mathfrak{k} : (\mathfrak{d}_0, \mathfrak{g}_0, \mathfrak{h}_0)_{\beta_0} \dashrightarrow (\mathfrak{d}_1, \mathfrak{g}_1, \mathfrak{h}_1)_{\beta_1}$ over $\Phi : H_0 \rightarrow H_1$. Finally, we verify that the image of $\mathfrak{k} \ltimes \text{ann}(\mathfrak{k})$ under the projection $\mathfrak{g} \ltimes \mathfrak{d}_\beta^* \rightarrow \mathfrak{q}$ is equal to \mathfrak{l} . Since $f(\mathfrak{l}) \subseteq \mathfrak{k}$, we have $f^*(\text{ann}(\mathfrak{k})) \subseteq \mathfrak{l}^\perp = \mathfrak{l}$. Since the projection coincides with f^* on \mathfrak{d}^* , this shows that the image of $\text{ann}(\mathfrak{k})$ is contained in \mathfrak{l} . On the other hand, the image of $\mathfrak{k} \cap (\mathfrak{g} \ltimes \mathfrak{d}_\beta^*)$ equals the image of $\text{gr}(\psi)^\top \subseteq \mathfrak{l}$. This shows that the reduced Lagrangian is contained in \mathfrak{l} , and hence coincides with \mathfrak{l} . \square

The proof used:

Lemma 4.8. *Suppose $\mathfrak{l} : V \rightarrow V'$ is a linear map taking $\beta \in S^2V$ to $\beta' \in S^2V'$. Suppose $W \subseteq V$, $W' \subseteq V'$ with $\mathfrak{l}(W) \subseteq W'$. If $W \subseteq V$ is β -coisotropic, then W' is β' -coisotropic.*

Proof. Indeed, $\mathfrak{l}(W) \subseteq W'$ implies $\mathfrak{l}^*(\text{ann}(W')) \subseteq \text{ann}(W)$. Thus, if β vanishes on $\text{ann}(W)$ then $\beta' = \mathfrak{l}(\beta)$ vanishes on $\text{ann}(W')$. \square

5. EXPLICIT FORMULAS

5.1. The \mathcal{CA} -groupoid structure in terms of the trivialization. Let (\mathbb{A}, E) be a Dirac Lie group structure on H . In this Section we will work out the formulas for the Courant algebroid structure and \mathcal{VB} -groupoid structure on \mathbb{A} in terms of the trivialization $\mathbb{A} \xrightarrow{\cong} H \times \mathfrak{q}$, obtained in Proposition 3.9.

We will need some background from the theory of matched pairs. Given a Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, one obtains actions of \mathfrak{h} on $\mathfrak{g} \cong \mathfrak{d}/\mathfrak{h}$ and \mathfrak{g} on $\mathfrak{h} \cong \mathfrak{d}/\mathfrak{g}$, satisfying the compatibility conditions of a *matched pair* $\mathfrak{g} \bowtie \mathfrak{h}$ of Lie algebras. Moreover, letting $p_{\mathfrak{h}} \in \text{End}(\mathfrak{d})$ be the projection to \mathfrak{h} along \mathfrak{g} , the \mathfrak{g} -action on \mathfrak{h} extends to a \mathfrak{d} -action on \mathfrak{h} by

$$\nu \mapsto p_{\mathfrak{h}}([\xi, \nu]), \quad \xi \in \mathfrak{d}, \quad \nu \in \mathfrak{h}.$$

Similarly, an H -equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ defines a linear action \bullet of the group H on $\mathfrak{g} = \mathfrak{d}/\mathfrak{h}$ and a Lie algebra action $\varrho : \mathfrak{g} \rightarrow \mathfrak{X}(H)$, satisfying the compatibility conditions

of a *matched pair* $\mathfrak{g} \bowtie H$ between a Lie group and a Lie algebra. These actions are given by

$$(23) \quad h \bullet \xi = (1 - p_{\mathfrak{h}}) \text{Ad}_h \xi, \quad h \in H, \quad \xi \in \mathfrak{g}$$

and $\iota(\varrho(\xi))\theta_h^R = p_{\mathfrak{h}}(\text{Ad}_h \xi)$ for $h \in H$, $\xi \in \mathfrak{g}$. Furthermore, the \mathfrak{g} -action on H combines with the \mathfrak{h} -action $\nu \rightarrow \nu^L$ to a Lie algebra action $\varrho: \mathfrak{d} \rightarrow \mathfrak{X}(H)$, by the same formula:

$$(24) \quad \iota(\varrho(\zeta))\theta_h^R = p_{\mathfrak{h}}(\text{Ad}_h \zeta), \quad h \in H, \quad \zeta \in \mathfrak{d}.$$

See Appendix B for more details. One has the following extension of Proposition 3.4.

Proposition 5.1. *There is a 1-1 correspondence between*

- (i) *Vacant \mathcal{LA} -groupoids $E \rightrightarrows \mathfrak{g}$ over groups $H \rightrightarrows \text{pt}$,*
- (ii) *H -equivariant Lie algebra triples $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, and*
- (iii) *Matched pairs $\mathfrak{g} \bowtie H$.*

Furthermore, if $x \in E_h$ with $s(x) = \xi$ then $t(x) = h \bullet \xi$, $a(x) = \varrho(\xi)_h$.

Here, the equivalence (i) \Leftrightarrow (iii) was observed by Mackenzie [22, 23]. Again, further details are given in Appendix B.

Let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ and $f: \mathfrak{q} \rightarrow \mathfrak{g}$ be as in Section 3.2. The action \bullet of H on \mathfrak{g} defines an action on the dual space $\mathfrak{g}^* \cong \mathfrak{r}^\perp \subseteq \mathfrak{q}$, which we again denote by \bullet . Recalling that f^* takes the Ad_h -invariant subspace $\mathfrak{g}^* \cong \text{ann}(\mathfrak{h})$ isomorphically to \mathfrak{r}^\perp , this action is characterized by

$$(25) \quad h \bullet f^*(\mu) = f^*(\text{Ad}_h \mu), \quad h \in H, \quad \mu \in \text{ann}(\mathfrak{h}).$$

The restriction of the metric on \mathfrak{q} to the subspace \mathfrak{r}^\perp is invariant under the H -action:

$$(26) \quad \langle h \bullet \nu, h \bullet \nu' \rangle = \langle \nu, \nu' \rangle, \quad \nu, \nu' \in \mathfrak{r}^\perp, \quad h \in H.$$

This follows by writing $\nu = f^*(\mu)$, $\nu' = f^*(\mu')$ and using the H -equivariance of $\beta^\sharp = f \circ f^*$. Write $p_{\mathfrak{r}} \in \text{End}(\mathfrak{q})$ for the projection to \mathfrak{r} along \mathfrak{g} . We are in a position to give explicit structural formulas for Dirac Lie groups.

Theorem 5.2. *The Dirac Lie group structure defined by the H -equivariant Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ is given by*

$$(\mathbb{A}, E) = (H \times \mathfrak{q}, H \times \mathfrak{g}).$$

Here \mathbb{A} carries the structure of an action Courant algebroid, for the action $\varrho^{\mathbb{A}} = \varrho \circ f: \mathfrak{q} \rightarrow \mathfrak{X}(H)$,

$$(27) \quad \iota(\varrho^{\mathbb{A}}(\zeta))\theta_h^R = p_{\mathfrak{h}}(\text{Ad}_h f(\zeta)).$$

The \mathcal{VB} -groupoid structure has source and target maps $s, t: \mathbb{A} \rightarrow \mathfrak{g}$,

$$s(h, \zeta) = p_{\mathfrak{r}}^*(\zeta), \quad t(h, \zeta) = h \bullet (1 - p_{\mathfrak{r}})(\zeta);$$

the inclusion of units is the map $\mathfrak{g} \rightarrow \mathbb{A}$, $\xi \mapsto (e, \xi)$, and the multiplication of composable elements is given by

$$(h_1, \zeta_1) \circ (h_2, \zeta_2) = (h_1 h_2, \zeta_2 + h_1^{-1} \bullet (1 - p_{\mathfrak{r}}^*)\zeta_1).$$

Proof. Let $\varrho^{\mathbb{A}}: \mathfrak{q} \rightarrow \mathfrak{X}(H)$ be the Lie algebra action described in Proposition 3.9. By definition, $\varrho^{\mathbb{A}}(\zeta) = a(\sigma)$ where $\sigma \in \Gamma(\mathbb{A})$ is the constant section corresponding to ζ (i.e. $\sigma \sim_T \zeta$). For $\zeta = \tau \in \mathfrak{r}$, the constant section σ takes values in $\ker(t)$, hence $\sigma_h = 0_h \circ \sigma_e$ for all h . Since the anchor a is a groupoid homomorphism, it follows that $\varrho^{\mathbb{A}}(\tau)_h = 0_h \circ \varrho^{\mathbb{A}}(\tau)_0$. Equivalently,

$\varrho^q(\tau)$ is left-invariant. Hence $\iota(\varrho^q(\tau))\theta_h^R = \text{Ad}_h(\varrho^q(\tau)_e) = \text{Ad}_h(f(\tau)) = p(\text{Ad}_h(f(\tau)))$, i.e. $\varrho^q(\tau) = \varrho(f(\tau))$. On the other hand, Equation (23) shows $\iota(\varrho(\xi))\theta_h^R = p_{\mathfrak{h}}(\text{Ad}_h(f(\xi)))$ for $\xi \in \mathfrak{g}$, hence $\varrho^q(\xi) = \varrho(f(\xi))$. This proves $\varrho^q = \varrho \circ f$.

We next consider the groupoid structure. Use the trivialization to write elements of \mathbb{A} in the form $x = (h, \zeta)$. Recall that on the vacant \mathcal{VB} -subgroupoid E , the trivialization is given by the source map. Hence, by Proposition 5.1 we have

$$(28) \quad s(h, \xi) = \xi, \quad t(h, \xi) = h \bullet \xi$$

for $(h, \xi) \in E$, and

$$(29) \quad (h_1, \xi_1) \circ (h_2, \xi_2) = (h_1 h_2, \xi_2)$$

for $(h_i, \xi_i) \in E$ with $h_2 \bullet \xi_2 = \xi_1$. Consider now a general element $(h, \zeta) \in \mathbb{A}$. By definition of the trivialization,

$$(30) \quad (h, \zeta) = j(h, \zeta) \circ (e, \zeta).$$

Since $s(j(h, \zeta)) = t(e, \zeta) = (1 - p_{\mathfrak{r}})\zeta$ by definition of $p_{\mathfrak{r}}$, it follows that $j(h, \zeta) = (h, (1 - p_{\mathfrak{r}})\zeta)$. We conclude that $t(h, \zeta) = t(j(h, \zeta)) = h \bullet (1 - p_{\mathfrak{r}})\zeta$, and $s(h, \zeta) = s(e, \zeta) = p_{\mathfrak{r}}^*\zeta$.

To find the groupoid multiplication, consider first a product $(h_1, 0) \circ (h_2, \nu)$ with $(h_2, \nu) \in \ker(t)$, i.e. $\nu \in \mathfrak{r}$. The product lies in $\ker(t)$, hence it is of the form $(h_1, 0) \circ (h_2, \nu) = (h_1 h_2, \nu')$ for some $\nu' \in \mathfrak{r}$. Taking inner products with the identity $(h_1, h_2 \bullet \xi) \circ (h_2, \xi) = (h_1 h_2, \xi)$ for $\xi \in \mathfrak{g}$, we obtain $\langle 0, h_2 \bullet \xi \rangle + \langle \nu, \xi \rangle = \langle \nu', \xi \rangle$, hence $\nu' = \nu$. Thus

$$(h_1, 0) \circ (h_2, \nu) = (h_1 h_2, \nu), \quad \nu \in \mathfrak{r}.$$

Similarly, consider a product $(h_1, \tau) \circ (h_2, 0) = (h_1 h_2, \tau')$ with $(h_1, \tau) \in \ker(s)$, thus $\tau \in \mathfrak{r}^\perp$. Then $\tau' \in \mathfrak{r}^\perp$, and $\langle \tau, h_2 \bullet \xi \rangle + \langle 0, \xi \rangle = \langle \tau', \xi \rangle$ for $\xi \in \mathfrak{g}$, proving $\tau' = (h_2)^{-1} \bullet \tau$.

For a general product $(h_1, \zeta_1) \circ (h_2, \zeta_2)$ of composable elements, write $\zeta_2 = (1 - p_{\mathfrak{r}})\zeta_2 + p_{\mathfrak{r}}\zeta_2$ and $\zeta_1 = p_{\mathfrak{r}}^*\zeta_1 + (1 - p_{\mathfrak{r}}^*)\zeta_1$. We obtain

$$\begin{aligned} (h_1, \zeta_1) \circ (h_2, \zeta_2) &= (h_1, 0) \circ (h_2, p_{\mathfrak{r}}\zeta_2) + (h_1, (1 - p_{\mathfrak{r}}^*)\zeta_1) \circ (h_2, 0) + (h_1, p_{\mathfrak{r}}^*\zeta_1) \circ (h_2, (1 - p_{\mathfrak{r}})\zeta_2) \\ &= (h_1 h_2, p_{\mathfrak{r}}\zeta_2 + h_1^{-1} \bullet (1 - p_{\mathfrak{r}}^*)\zeta_1 + (1 - p_{\mathfrak{r}})\zeta_2) \\ &= (h_1 h_2, \zeta_2 + h_1^{-1} \bullet (1 - p_{\mathfrak{r}}^*)\zeta_1). \end{aligned}$$

□

5.2. Examples.

5.2.1. The standard Dirac Lie group structure. For any Lie group H , we have the H -equivariant Dirac Manin triple $(\mathfrak{h} \ltimes \mathfrak{h}^*, \mathfrak{h}^*, \mathfrak{h})_\beta$, with β the symmetric bilinear form given by the pairing. Since β is non-degenerate and $\mathfrak{g} = \mathfrak{h}^*$ is Lagrangian, we have (cf. Example 3.7) $\mathfrak{q} = \mathfrak{d}$, with f the identity. The projections $p_{\mathfrak{h}}$ and $(1 - p_{\mathfrak{h}}^*)$ coincide, and our formulas specialize to

$$\begin{aligned} s(h, \nu, \mu) &= \mu, \quad t(h, \nu, \mu) = \text{Ad}_h \mu, \\ (h_1, \nu_1, \mu_1) \circ (h_2, \nu_2, \mu_2) &= (h_1 h_2, \nu_2 + \text{Ad}_{h_1^{-1}} \nu_1, \mu_2). \end{aligned}$$

The action of $\mathfrak{h} \ltimes \mathfrak{h}^*$ on H is given by the left-invariant vector fields, $\varrho(\nu, \mu) = \nu^L$. This is the standard Dirac Lie group structure $(\mathbb{A}, E) = (\mathbb{T}H, T^*H)$, written in left-trivialization.

5.2.2. *The Cartan-Dirac structure.* Given a Lie group G with an invariant metric on \mathfrak{g} , one can form the Dirac Manin triple $(\bar{\mathfrak{g}} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, 0 \oplus \mathfrak{g})_\beta$ where β is given by the metric on $\bar{\mathfrak{g}} \oplus \mathfrak{g}$, and \mathfrak{g}_Δ is the diagonal. Again $\mathfrak{q} = \mathfrak{d}$, $f = \text{id}$. For $h \in H = \{1\} \times G$ we have $\text{Ad}_h(\xi, \xi') = (\xi, \text{Ad}_h \xi')$. It follows that the action \bullet on \mathfrak{g} (hence also on \mathfrak{r}^\perp) is the trivial action:

$$h \bullet (\xi, \xi) = (1 - p_{\mathfrak{h}})(\xi, \text{Ad}_h \xi) = (\xi, \xi).$$

The formulas for the groupoid structure simplify to

$$s(h, \xi, \xi') = \xi', \quad t(h, \xi, \xi') = \xi, \quad (h_1, \xi_1, \xi'_1) \circ (h_2, \xi_2, \xi'_2) = (h_1 h_2, \xi_1, \xi'_2).$$

From $\iota(\varrho(\xi, \xi'))\theta_h^R = p_{\mathfrak{h}} \text{Ad}_h(\xi, \xi') = p_{\mathfrak{h}}(\xi, \text{Ad}_h \xi') = \text{Ad}_h \xi' - \xi$ we obtain

$$\varrho(\xi, \xi') = (\xi')^L - \xi^R.$$

The resulting Dirac Lie group structure $(\mathbb{A}, E) = (G \times (\bar{\mathfrak{g}} \oplus \mathfrak{g}), G \times \mathfrak{g}_\Delta)$ is the Cartan-Dirac structure from Example 2.12.

5.2.3. *Dirac Lie group structures over $H = \text{pt}$.* If the group H is trivial, then the Dirac Manin triple is of the form $(\mathfrak{d}, \mathfrak{d}, 0)_\beta$. Dirac Lie group structures over pt are hence classified by Lie algebras \mathfrak{d} with invariant elements $\beta \in S^2 \mathfrak{d}$. (The same data also classify the \mathcal{CA} -groupoid structures \mathbb{A} over $H = \text{pt}$, since \mathbb{A} extends uniquely to a Dirac Lie group structure by putting $E = \mathbb{A}^{(0)}$.) We find

$$(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma = (\mathfrak{d} \ltimes \mathfrak{d}_\beta^*, \mathfrak{d}, (\mathfrak{d}_\beta^*)^\perp)_{\hat{\beta}}$$

with f the projection along $(\mathfrak{d}_\beta^*)^\perp$. The \mathcal{VB} -groupoid structure on $\mathfrak{d} \ltimes_\beta \mathfrak{d}_\beta^* \rightrightarrows \mathfrak{d}$ is given by $s(\xi, \mu) = \xi$, $t(\xi, \mu) = \xi + \beta^\sharp(\mu)$, and the groupoid multiplication of composable elements is given by

$$(\xi_1, \mu_1) \circ (\xi_2, \mu_2) = (\xi_2, \mu_1 + \mu_2).$$

Note that \mathfrak{d} is a subgroupoid, as required.

We can also classify the multiplicative Main pairs over $H = \text{pt}$. First, by Drinfel'd's classification of \mathcal{CA} -groupoids over $H = \text{pt}$ (see Remark 3.6), we may assume that $\mathbb{A} = \mathfrak{d} \ltimes \mathfrak{d}_\beta^*$. Suppose E is a multiplicative Dirac structure inside $\mathfrak{d} \ltimes \mathfrak{d}_\beta^*$, and let $\mathfrak{g} := E^{(0)}$. Then $E = E^\perp \subseteq \mathfrak{g}^\perp = \mathfrak{d} \oplus \text{ann}(\mathfrak{g})$. The kernel of the source map $E \rightarrow \mathfrak{g}$ has dimension $\dim E - \dim \mathfrak{g} = \dim \mathfrak{d} - \dim \mathfrak{g} = \dim \text{ann}(\mathfrak{g})$, and is contained in $\mathfrak{r} = \mathfrak{d}_\beta^*$. This shows $\text{ann}(\mathfrak{g}) \subseteq E$. Thus $E = \mathfrak{g} \oplus \text{ann}(\mathfrak{g})$ as a vector space. The form β vanishes on $\text{ann}(\mathfrak{g})$ since E is isotropic. Finally, E is a Lie subalgebra of \mathbb{A} if and only if \mathfrak{g} is a Lie subalgebra of \mathfrak{d} . Thus $E = \mathfrak{g} \ltimes \text{ann}(\mathfrak{g})$ as a Lie algebra. Conversely, for any β -coisotropic Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{d}$, the semi-direct product $E = \mathfrak{g} \ltimes \text{ann}(\mathfrak{g})$ defines a Lagrangian Lie subalgebra. We have $E \cap \mathfrak{d}^* = E \cap (\mathfrak{d}^*)^\perp = \text{ann}(\mathfrak{g})$, hence the restriction of both s, t to E are the projections to \mathfrak{g} along $\text{ann}(\mathfrak{g})$. It is then clear that E is a subgroupoid. We have shown:

Proposition 5.3. *There is a 1-1 correspondence between*

- (i) *Multiplicative Manin pairs (\mathbb{A}, E) over $H = \text{pt}$,*
- (ii) *Dirac Manin pairs $(\mathfrak{d}, \mathfrak{g})_\beta$ (i.e. \mathfrak{g} is a β -coisotropic Lie subalgebra of \mathfrak{d}).*

6. THE LAGRANGIAN COMPLEMENT F

Let (\mathbb{A}, E) be a Dirac Lie group structure on H . We will show that E has a distinguished Lagrangian complement. The splitting $\mathbb{A} = E \oplus F$ defines a bi-vector field π_H , and (H, π_H) is a quasi-Poisson \mathfrak{g} -space in the sense of Alekseev and Kosmann-Schwarzbach [2].

6.1. Quasi-Poisson \mathfrak{g} -manifolds. A *quasi-Manin triple* $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})_\gamma$ is Lie algebra \mathfrak{q} with a non-degenerate element $\gamma \in (S^2\mathfrak{q})^\mathfrak{q}$, together with a Lagrangian Lie subalgebra \mathfrak{g} and a Lagrangian subspace \mathfrak{n} complementary to \mathfrak{g} . The quasi-Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})_\gamma$ determines a trivector $\chi \in \wedge^3\mathfrak{g} \subseteq \wedge^3\mathfrak{q}$ by the equation

$$\chi(\zeta, \zeta', \zeta'') = \langle [\zeta, \zeta'], \zeta'' \rangle, \quad \zeta, \zeta', \zeta'' \in \mathfrak{n},$$

as well as a *cobracket*

$$\partial: \mathfrak{g} \rightarrow \wedge^2\mathfrak{g}, \quad \partial(\xi)(\zeta, \zeta') = \langle [\zeta, \zeta'], \xi \rangle, \quad \zeta, \zeta' \in \mathfrak{n}, \quad \xi \in \mathfrak{g}.$$

Here \mathfrak{g} is identified with \mathfrak{n}^* . Note that χ measures the failure of \mathfrak{n} to define a Lie subalgebra of \mathfrak{q} . A *quasi-Poisson \mathfrak{g} -space* [2] for the quasi-Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})_\gamma$ is a manifold M with an action $\varrho_M: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ and a bivector field $\pi_M \in \mathfrak{X}^2(M)$ satisfying

$$(31) \quad \begin{aligned} \frac{1}{2}[\pi_M, \pi_M] &= \varrho_M(\chi), \\ \mathcal{L}_{\varrho_M(\xi)}\pi_M &= \varrho_M(\partial\xi), \quad \xi \in \mathfrak{g}. \end{aligned}$$

As shown in [5], this definition is equivalent to a morphism of Manin pairs,

$$(32) \quad K: (\mathbb{T}M, TM) \dashrightarrow (\mathfrak{q}, \mathfrak{g}).$$

Here K determines the \mathfrak{g} -action $\varrho_M: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ by the condition $\varrho_M(\xi) \sim_K \xi$, $\xi \in \mathfrak{g}$, and the bivector field π_M on M is described in terms of its graph as $\text{Gr}(\pi_M) = \mathfrak{n} \circ K \subseteq \mathbb{T}M$, the ‘backward image’ of \mathfrak{n} . More generally, any morphism of Manin pairs $R: (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g})$ determines a quasi-Poisson structure on M , by taking its composition with the morphism $(\mathbb{T}M, TM) \dashrightarrow (\mathbb{A}, E)$ from Example 1.6. Here the \mathfrak{g} -action is $\varrho_M(\xi) = \mathfrak{a}(e(\xi))$ where $\mathfrak{g} \rightarrow \Gamma(E)$, $\xi \mapsto e(\xi)$ is defined by the condition $e(\xi) \sim_R \xi$. The bi-vector field π_M is determined by the splitting $\mathbb{A} = E \oplus F$, and is locally given by the formula $\pi_M = \frac{1}{2}\mathfrak{a}(e_i) \wedge \mathfrak{a}(f^i)$ where e_i, f^j are sections of E, F with $\langle e_i, f^j \rangle = \delta_i^j$. (See e.g. [19, Theorem 3.16]).

6.2. Quasi-Poisson structures from Dirac Lie groups. Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ be an H -equivariant Dirac Manin triple, and let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ be the Dirac Manin triple constructed from it. The following standard procedure turns the Lie algebra complement \mathfrak{r} into a *Lagrangian* complement. As before we denote by $p_{\mathfrak{r}} \in \text{End}(\mathfrak{q})$ the projection to \mathfrak{r} along \mathfrak{g} , so that $1 - p_{\mathfrak{r}}$ and $p_{\mathfrak{r}}^*$ are the projections to \mathfrak{g} along $\mathfrak{r}, \mathfrak{r}^\perp$, respectively. Their average $\frac{1}{2}((1 - p_{\mathfrak{r}}) + p_{\mathfrak{r}}^*)$ is again a projection to \mathfrak{g} , and its kernel \mathfrak{n} is the desired Lagrangian complement. Thus \mathfrak{n} is the mid-point between $\mathfrak{r}, \mathfrak{r}^\perp$ in the affine space of complements to \mathfrak{g} . If ϵ_i is a basis of \mathfrak{g} , and ϕ^i a basis of \mathfrak{r}^\perp with $\langle \epsilon_i, \phi^j \rangle = \delta_i^j$, the space \mathfrak{n} has basis

$$\nu^i = \phi^i - \frac{1}{2} \sum_j \langle \phi^i, \phi^j \rangle \epsilon_j.$$

Note that the ‘r-matrix’

$$\frac{1}{2} \sum_i \epsilon_i \wedge \nu^i = \frac{1}{2} \sum_i \epsilon_i \wedge \phi^i \in \wedge^2\mathfrak{q}$$

is independent of the choice of basis. Letting $\epsilon^i \in \mathfrak{g}^*$ be the dual basis, we have $\phi^i = f^*(\epsilon^i)$, and using $\langle f^*(\epsilon^i), f^*(\epsilon^j) \rangle = \beta(\epsilon^i, \epsilon^j)$ we obtain

$$(33) \quad \nu^i = f^*(\epsilon^i) - \frac{1}{2} \sum_j \beta(\epsilon^i, \epsilon^j) \epsilon_j.$$

As explained in the previous section, the ‘trivializing morphism’ $T: (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g})$ gives H the structure of a quasi-Poisson space for the quasi-Manin triple $(\mathfrak{q}, \mathfrak{g}, \mathfrak{n})$. In terms of the trivialization $\mathbb{A} = H \times \mathfrak{q}$, the Lagrangian complement $F = \mathfrak{n} \circ T$ is simply the trivial bundle $H \times \mathfrak{n}$.

Proposition 6.1. *In the affine space of Lagrangian complements to E in \mathbb{A} , the sub-bundle F is the mid-point between $\ker(s)$ and $\ker(t)$. One has,*

$$(34) \quad F = \{x \in \mathbb{A} \mid h \bullet s(x) + t(x) = 0\}$$

where $h \in H$ indicates the base point of x .

Note that E is similarly given by a condition $h \bullet s(x) - t(x) = 0$.

Proof. The first claim follows since the trivialization of \mathbb{A} restricts to isomorphisms $E \cong H \times \mathfrak{g}$, $\ker(t) \cong H \times \mathfrak{r}$ and $\ker(s) \cong H \times \mathfrak{r}^\perp$. The second part follows from the characterization of \mathfrak{n} as the kernel of $\frac{1}{2}((1 - p_{\mathfrak{r}}) + p_{\mathfrak{r}}^*)$, since $h \bullet s(h, \zeta) = h \bullet p_{\mathfrak{r}}^*(\zeta)$ and $t(h, \zeta) = h \bullet (1 - p_{\mathfrak{r}})(\zeta)$ in the trivialization. \square

Since \mathfrak{a} is given on constant sections of $\mathbb{A} = H \times \mathfrak{q}$ by the action map ϱ , we obtain:

Proposition 6.2. *For any Dirac Lie group structure (\mathbb{A}, E) on H , with corresponding \mathfrak{q} -action $\varrho: \mathfrak{q} \rightarrow \mathfrak{X}(H)$, one obtains a quasi-Poisson structure on H , with bivector field*

$$\pi_H = \frac{1}{2} \sum_i \varrho(\epsilon_i) \wedge \varrho(f^*(\epsilon^i)).$$

and \mathfrak{g} -action $\varrho_H = \varrho|_{\mathfrak{g}}$.

6.3. Multiplicative properties. We next consider the multiplicative aspects of the quasi-Poisson structure. The composition of morphisms

$$(\mathbb{A}, E) \times (\mathbb{A}, E) \dashrightarrow (\mathbb{A}, E) \dashrightarrow (\mathfrak{q}, \mathfrak{g})$$

gives $H \times H$ the structure of a quasi-Poisson \mathfrak{g} -space $(H \times H, \pi_{H \times H})$, with the property that the underlying map $\text{Mult}_H: H \times H \rightarrow H$ is a morphism of quasi-Poisson manifolds. The \mathfrak{g} -action $\varrho_{H \times H}$ is computed as follows. Using the trivialization $E = H \times \mathfrak{g}$, the equality $(h_1, \xi_1) \circ (h_2, \xi_2) = (h_1 h_2, \xi)$ holds if and only if $\xi_2 = \xi$, $\xi_1 = h_2 \bullet \xi$. Thus

$$\varrho_{H \times H}(\xi)_{(h_1, h_2)} = (\varrho_H(h_2 \bullet \xi)_{h_1}, \varrho_H(\xi)_{h_2}).$$

Proposition B.2 confirms that the multiplication in H is equivariant for this twisted action. (More generally this holds true for any matched pair between a Lie group and a Lie algebra.)

The bivector field $\pi_{H \times H}$ is determined by the splitting $(E \times E) \oplus F'$, where F' is the backward image $F' = \mathfrak{n} \circ (T \circ \text{Mult}_{\mathbb{A}}) = F \circ \text{Mult}_{\mathbb{A}} = \{(x_1, x_2) \in \mathbb{A} \times \mathbb{A} \mid x_1 \circ x_2 \in F\}$. Thus

$$F' = \{(x_1, x_2) \in \mathbb{A} \times \mathbb{A} \mid s(x_1) = t(x_2), \quad h_1 h_2 \bullet s(x_2) + t(x_1) = 0\}.$$

Since F' and $F \times F$ are both Lagrangian complements to $E \times E$, there is a unique section $\lambda \in \Gamma(\wedge^2(E \times E))$ with the property that

$$F' = (\text{id} + \lambda^\sharp)(F \times F),$$

where $\lambda^\sharp: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A} \times \mathbb{A}$ is the bundle map defined by λ , and

$$\pi_{H \times H} = \pi_H^{(1)} + \pi_H^{(2)} + \mathbf{a}(\lambda)$$

where $\pi_H^{(i)}$, $i = 1, 2$ is the bivector field π_H on the i -th factor of $H \times H$, and \mathbf{a} is the anchor map for $\mathbb{A} \times \mathbb{A}$. (See [1, Proposition 1.18].) It remains to compute λ .

Proposition 6.3. *The section $\lambda \in \Gamma(\wedge^2(E \times E))$ is given in terms of the trivialization $E = H \times \mathfrak{g}$ as*

$$\lambda = -\frac{1}{2} \sum_{ij} \beta(\epsilon^i, \epsilon^j) (\epsilon_i, 0) \wedge (0, h_2^{-1} \bullet \epsilon_j).$$

Thus,

$$\mathbf{a}(\lambda) = -\frac{1}{2} \sum_{ij} \beta(\epsilon^i, \text{Ad}_{h_2} \epsilon^j) \varrho_H(\epsilon_i)^{(1)} \wedge \varrho_H(\epsilon_j)^{(2)} \in \mathfrak{X}(H \times H),$$

where the superscripts (1), (2) indicate the vector fields operating on the first resp. second H -factor.

Proof. We will use the trivialization $\mathbb{A} = H \times \mathfrak{g}$, and omit base points to simplify notation. For all $(\tau_1, \tau_2) \in F \times F$ at a given base point h_1, h_2 , there is a unique element $(\xi_1, \xi_2) \in E \times E$ such that $(\tau_1 + \xi_1, \tau_2 + \xi_2) \in F'$. Thus, $(\tau_1 + \xi_1) \circ (\tau_2 + \xi_2) \in F$, i.e.

$$s(\tau_1 + \xi_1) = t(\tau_2 + \xi_2), \quad t(\tau_1 + \xi_1) = -h_1 h_2 \bullet s(\tau_2 + \xi_2)$$

Using $t(\xi_i) = h_i \bullet s(\xi_i)$ and $t(\tau_i) = -h_i \bullet s(\tau_i)$, and solving for $\xi_i = s(\xi_i)$, we find

$$\xi_1 = -h_2 \bullet s(\tau_2), \quad \xi_2 = h_2^{-1} \bullet s(\tau_1).$$

This shows that λ is of the form $\lambda = \sum_i (\epsilon_i, 0) \wedge (0, s^i)$ for some $s^i \in \mathfrak{g}$ (depending on h_1, h_2). Taking $\tau_2 = 0$ and $\tau_1 = \nu^i$ the basis element of \mathfrak{n} , we find

$$(0, s^i) = \lambda^\sharp(\nu^i, 0) = (0, h_2^{-1} \bullet s(\nu^i)) = -\frac{1}{2} \sum_l \beta(\epsilon^i, \epsilon^l) (0, h_2^{-1} \bullet \epsilon_l). \quad \square$$

Example 6.4. Let us specialize the formulas to the Cartan-Dirac structure from Section 5.2.2, given by the G -invariant Dirac Manin triple $(\bar{\mathfrak{g}} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, 0 \oplus \mathfrak{g})_\beta$. In this case $\mathfrak{q} = \mathfrak{d}$, $\mathfrak{r} = \mathfrak{h}$. We have $\mathfrak{r}^\perp = \mathfrak{g} \oplus 0$, and $\mathfrak{n} = \{(-\xi, \xi) | \xi \in \mathfrak{g}\}$ is the anti-diagonal. Letting e_i be a basis of \mathfrak{g} , with B -dual basis e^i , the corresponding basis of \mathfrak{g}_Δ is $\epsilon_i = (e_i, e_i)$, hence $f^*(\epsilon^i) = (-e^i, 0) \in \mathfrak{r}^\perp$, and the dual basis of \mathfrak{n} is $\nu^i = \frac{1}{2}(-e^i, e^i)$. The resulting bivector field on G is

$$\pi_G = \frac{1}{2} \sum_i \varrho(e_i, e_i) \wedge \varrho(-e^i, 0) = \frac{1}{2} \sum_i ((e_i)^L - (e_i)^R) \wedge (e^i)^R = \frac{1}{2} \sum_i (e_i)^L \wedge (e^i)^R.$$

Since the action \bullet is trivial, and $\beta(\epsilon^i, \epsilon^j) = -B(e^i, e^j)$, the section $\lambda \in \Gamma(\wedge^2(E \times E))$ is given by the formula $\lambda = \frac{1}{2} \sum_i (e_i, e_i)^{(1)} \wedge (e^i, e^i)^{(2)}$.

7. EXACT DIRAC LIE GROUPS

A Dirac Lie group structure (\mathbb{A}, E) on H is called *exact* if the underlying Courant algebroid \mathbb{A} is exact (cf. Section 1). We will show that in this case, \mathbb{A} has a distinguished isotropic splitting, giving an identification $\mathbb{A} \cong \mathbb{T}H_\eta$ for a suitable closed 3-form $\eta \in \Omega^3(H)$.

7.1. Characterization in terms of Dirac Manin triples. Exact Dirac Lie group structures have the following characterization in terms of the associated Dirac Manin triples.

Proposition 7.1. *Let (\mathbb{A}, E) be a Dirac Lie group structure on H , with corresponding Dirac Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. Then the Dirac Lie group structure is exact if and only if β is non-degenerate and \mathfrak{g} is Lagrangian with respect to β .*

Proof. Let $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma$ be the Dirac Manin triple constructed from $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$.

\Rightarrow . Suppose \mathbb{A} is exact. Then $\mathfrak{a}_e: \mathfrak{q} = \mathbb{A}_e \rightarrow \mathfrak{h} = T_e H$ is surjective, with kernel \mathfrak{g} . It follows that \mathfrak{a}_e restricts to an isomorphism $\mathfrak{r} \rightarrow \mathfrak{h}$; hence $f: \mathfrak{q} \rightarrow \mathfrak{d}$ is an isomorphism. Since $f(\gamma) = \beta$, we conclude that β is non-degenerate and \mathfrak{g} is Lagrangian.

\Leftarrow . If β is non-degenerate and \mathfrak{g} is Lagrangian, then (cf. Example 3.7) the map $f: \mathfrak{q} \rightarrow \mathfrak{d}$ gives an isomorphism $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma \cong (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$. Hence the action $\varrho^{\mathfrak{q}} = \varrho \circ f$ of \mathfrak{q} on H is transitive. Hence $\mathfrak{a}: \mathbb{A} \rightarrow TH$ is surjective, and by dimension count its kernel is $\mathfrak{a}^*(T^*H)$. \square

For the remainder of this section, we assume (\mathbb{A}, E) is an exact Dirac Lie group structure, so that β is non-degenerate, and \mathfrak{g} is Lagrangian in \mathfrak{d} . Using the isomorphism $(\mathfrak{q}, \mathfrak{g}, \mathfrak{r})_\gamma \cong (\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_\beta$ we will write $\mathfrak{d}, \mathfrak{h}, \beta$ in place of $\mathfrak{q}, \mathfrak{r}, \gamma$ and we omit the letter f . We will write $p \in \text{End}(\mathfrak{d})$ for the projection to \mathfrak{h} along \mathfrak{g} .

7.2. Symmetries.

Proposition 7.2. *Let (\mathbb{A}, E) be an exact Dirac Lie group structure on H . The action of $H \times H$ on H , given by $(h_1, h_2).h = h_1 h h_2^{-1}$ lifts uniquely to an action on the vector bundle E , in such a way that:*

- (a) *The lifted action is by CA automorphisms: It preserves the metric $\langle \cdot, \cdot \rangle$ and the bracket $[\cdot, \cdot]$, and the anchor map is equivariant.*
- (b) *The lifted action is compatible with the groupoid structure, in the sense that the action map $(H \times H) \times \mathbb{A} \rightarrow \mathbb{A}$ is a groupoid homomorphism (using the pair groupoid structure $H \times H \rightrightarrows H$).*

The action of the diagonal $H \subseteq H \times H$ on $\ker(\mathfrak{a}_e) \cong \mathfrak{g}$ is given by $h.\xi = p^(\text{Ad}_h \xi)$.*

Note that this action of H on \mathfrak{g} is different from the action \bullet , in general. The proposition says in particular that $t((h_1, h_2).x) = h_1.t(x)$, $s((h_1, h_2).x) = h_2.s(x)$, and

$$((h_1, h_2).x) \circ ((h'_1, h'_2).x') = (h_1, h'_2).(x \circ x')$$

if $s(x) = t(x')$ and $h_2 = h'_1$.

Proof. Regard \mathbb{A} as the reduction C/C^\perp of $\mathbb{T}H \times (\bar{\mathfrak{d}} \oplus \mathfrak{d})$, as in Section 3.3. The lift of the $H \times H$ -action to $\mathbb{T}H$, given as the direct sum of the tangent and cotangent lifts, is by Courant algebroid automorphisms, and is compatible with the groupoid structure $\mathbb{T}H \rightrightarrows \mathfrak{h}^*$. Similarly the action on $\bar{\mathfrak{d}} \oplus \mathfrak{d}$, where the i -th factor of H acts by Ad on the i -th factor of \mathfrak{d} ,

is by Courant algebroid automorphisms, and is compatible with the pair groupoid structure $\bar{\mathfrak{d}} \oplus \mathfrak{d} \rightrightarrows \mathfrak{d}$. The sub-bundle $C \subset \mathbb{T}H \times (\bar{\mathfrak{d}} \oplus \mathfrak{d})$ given by (11) is invariant under this action, hence so is C^\perp , and we obtain an induced action on $\mathbb{A} = C/C^\perp$ by Courant automorphisms compatible with the groupoid structure. It has the desired property that the anchor is equivariant and that $\ker(t)$ is invariant. In particular, the action of the diagonal $H \subseteq H \times H$ on $\ker(t_e) \cong T_e H = \mathfrak{h}$ is the adjoint action. Since the metric on $\mathbb{A}_e = \mathfrak{d}$ is preserved, we deduce the action on $\ker(\mathfrak{a}_e) = \mathfrak{g}$: For $\xi \in \mathfrak{g}$ and $\zeta \in \mathfrak{h}$,

$$\langle h.\xi, \zeta \rangle = \langle \xi, h^{-1}.\zeta \rangle = \langle \xi, \text{Ad}_{h^{-1}} \zeta \rangle = \langle \text{Ad}_h \xi, \zeta \rangle = \langle p^*(\text{Ad}_h \xi), \zeta \rangle,$$

thus $h.\xi = p^*(\text{Ad}_h \xi)$. For the uniqueness properties, suppose we are given any lift of the $H \times H$ -action to Courant automorphisms of \mathbb{A} , compatible with the groupoid structure. By the equivariance properties of the anchor and target maps, the action on $\ker(\mathfrak{a}) \cong T^*H$ and $\ker(t) \cong TH$ are uniquely determined, hence so is the action on $\mathbb{A} = \ker(\mathfrak{a}) \oplus \ker(t)$. \square

Remark 7.3. For possibly non-exact Dirac Lie group structures (\mathbb{A}, E) , a similar construction gives an action of $\mathfrak{r} \times \mathfrak{r}$ by infinitesimal Courant automorphisms of \mathbb{A} , where $\mathfrak{r} = \ker(t_e) \subseteq \mathfrak{d} = \mathbb{A}_e$.

Lemma 7.4. *Under the isomorphism $\mathbb{A} = H \times \mathfrak{d}$, the action of $\{e\} \times H \subset H \times H$ is given by $(e, h_2).(h, \zeta) = (hh_2^{-1}, \text{Ad}_{h_2} \zeta)$.*

Proof. Recall that $\mathbb{A} = C/C^\perp$ may be identified with the sub-bundle $C \cap (TH \times (\mathfrak{g} \oplus \mathfrak{d}))$. Under this identification, the trivialization is the projection to the last factor. Since $\{e\} \times H$ preserves this sub-bundle, the Lemma follows. \square

The description of the $H \times \{e\}$ -action in terms of the trivialization is more involved, but will not be needed here.

7.3. Splitting. We next show that the Courant algebroid \mathbb{A} admits a *canonical* $H \times H$ -invariant isotropic splitting $\mathfrak{l}: TH \rightarrow \mathbb{A}$. Recall that \mathfrak{n} denotes the Lagrangian complement to \mathfrak{g} in $\mathfrak{d} = \mathfrak{q}$, as described in Section 6.2.

Theorem 7.5. *Let (\mathbb{A}, E) be an exact Dirac Lie group structure on H . There is a unique $\{e\} \times H$ -equivariant Lagrangian splitting $\mathfrak{l}: TH \rightarrow \mathbb{A}$ such that $\mathfrak{l}(T_e H) = \mathfrak{n}$. In fact, the splitting is $H \times H$ -invariant. In terms of the trivialization $\mathbb{A} = H \times \mathfrak{d}$, the splitting is given on right-invariant vector fields by*

$$(35) \quad \mathfrak{l}(\nu^R) = (h, \text{Ad}_{h^{-1}}(\nu - \tfrac{1}{2}p^*(\nu))), \quad \nu \in \mathfrak{h}.$$

The splitting determines an $H \times H$ -equivariant isomorphism $\mathbb{A} \xrightarrow{\cong} \mathbb{T}H_\eta$, where η is the closed bi-invariant Cartan 3-form

$$\eta = \tfrac{1}{12} \langle \theta^R, [\theta^R, \theta^R] \rangle.$$

Proof. Let $\Pi: \mathbb{A} \rightarrow \mathbb{A}$ denote the projection to $\ker(t)$ along $\ker(\mathfrak{a})$. Since $\ker(t), \ker(\mathfrak{a})$ are $H \times H$ -invariant sub-bundles, the projection Π is $H \times H$ -equivariant. $1 - \Pi, \Pi^*$ are projections to $\ker(\mathfrak{a})$ with kernels $\ker(t), \ker(s)$ respectively. The kernel of the projection $\tfrac{1}{2}((1 - \Pi) + \Pi^*)$ is a Lagrangian sub-bundle complementary to $\ker(\mathfrak{a})$, defining an $H \times H$ -invariant isotropic splitting $\mathfrak{l}: TH \rightarrow \mathbb{A}$ having this sub-bundle as its range. At the group

unit, Π coincides with the projection $p: \mathfrak{d} = \mathbb{A}_e \rightarrow \mathfrak{h} = \ker(t_e)$ along $\mathfrak{g} = \ker(\mathfrak{a}_e)$, hence $l(T_e H)$ is the subspace $\mathfrak{n} = \ker((1-p) + p^*)$.

By Lemma 7.4, the right hand side of Formula (35) defines an $\{e\} \times H$ -splitting. To show that it equals the left hand side, it is hence enough to check at $h = e$. Since $\nu - \frac{1}{2}p^*(\nu)$ lies in $\mathfrak{n} = \ker((1-p) + p^*)$ and maps to ν under p , this proves (35).

It remains to compute the resulting 3-form, using the formula (2) for action Courant algebroids. For $\nu \in \mathfrak{h}$ let $\tilde{\nu} = \nu - \frac{1}{2}p^*(\nu)$ be its projection to \mathfrak{n} . Equation (2) gives

$$\begin{aligned} \langle \llbracket l(\nu_1^R), l(\nu_2^R) \rrbracket, l(\nu_3^R) \rangle &= \langle [\tilde{\nu}_1, \tilde{\nu}_2], \tilde{\nu}_3 \rangle + \langle \mathcal{L}(\nu_1^R) \text{Ad}_{h^{-1}} \tilde{\nu}_2, \tilde{\nu}_3 \rangle - \langle \mathcal{L}(\nu_2^R) \text{Ad}_{h^{-1}} \tilde{\nu}_3, \text{Ad}_{h^{-1}} \tilde{\nu}_3 \rangle \\ &\quad + \langle \mathcal{L}(\nu_3^R) \text{Ad}_{h^{-1}} \tilde{\nu}_1, \text{Ad}_{h^{-1}} \tilde{\nu}_3 \rangle \\ &= \langle [\tilde{\nu}_1, \tilde{\nu}_2], \tilde{\nu}_3 \rangle - \langle [\nu_1, \tilde{\nu}_2], \tilde{\nu}_3 \rangle \\ &\quad + \langle [\nu_2, \tilde{\nu}_1], \tilde{\nu}_3 \rangle - \langle [\nu_3, \tilde{\nu}_1], \tilde{\nu}_2 \rangle, \end{aligned}$$

where we used $\mathcal{L}(\nu^R) \text{Ad}_{h^{-1}} \zeta = -\text{Ad}_{h^{-1}}[\nu, \zeta]$ for $\nu \in \mathfrak{h}$, $\zeta \in \mathfrak{d}$. Using the definition $\tilde{\nu}_i = \nu_i - \frac{1}{2}p^*\nu_i$, note that terms with exactly two $p^*\nu_i$'s cancel out. The term with three $p^*\nu_i$'s is zero, since \mathfrak{g} is a Lagrangian Lie subalgebra. Terms with exactly one $p^*\nu_i$ simplify, e.g. $\langle [\nu_1, \nu_2], -\frac{1}{2}p^*\nu_3 \rangle = -\frac{1}{2}\langle p[\nu_1, \nu_2], \nu_3 \rangle = -\frac{1}{2}\langle [\nu_1, \nu_2], \nu_3 \rangle$. Denoting terms with two or more $p^*\nu_i$'s by \dots , we hence have

$$\langle [\tilde{\nu}_1, \tilde{\nu}_2], \tilde{\nu}_3 \rangle = \langle [\nu_1, \nu_2], \nu_3 \rangle - \frac{3}{2}\langle [\nu_1, \nu_2], \nu_3 \rangle + \dots = -\frac{1}{2}\langle [\nu_1, \nu_2], \nu_3 \rangle + \dots,$$

and similarly $\langle [\nu_1, \tilde{\nu}_2], \tilde{\nu}_3 \rangle = 0 + \dots$. We conclude

$$\langle \llbracket l(\nu_1^R), l(\nu_2^R) \rrbracket, l(\nu_3^R) \rangle = -\frac{1}{2}\langle [\nu_1, \nu_2], \nu_3 \rangle.$$

It follows that $\eta = \frac{1}{12}\langle \theta^R, [\theta^R, \theta^R] \rangle$. □

For later reference, note that the isomorphism $\mathbb{T}H_\eta \rightarrow \mathbb{A}$, $v + \alpha \mapsto l(v) + \mathfrak{a}^*(\alpha)$ is given in terms of the trivialization $\mathbb{A} = H \times \mathfrak{d}$ by the map $v + \alpha \mapsto (h, \zeta)$, with

$$(36) \quad \zeta = \text{Ad}_{h^{-1}}((1 - \frac{1}{2}p^*)\iota_v \theta_h^R + t(\alpha)), \quad \Leftrightarrow \quad \iota_v \theta_h^R = p(\text{Ad}_h \zeta), \quad \alpha = \langle \theta_h^R, \frac{1}{2}(p^* + (1-p))(\text{Ad}_h \zeta) \rangle.$$

7.4. Multiplicative properties. We next discuss the multiplicative aspects of the isomorphism $\mathbb{A} \cong \mathbb{T}H_\eta$. Let $\text{pr}_1, \text{pr}_2: H \times H \rightarrow H$ be the two projections, and let $\sigma \in \Omega^2(H \times H)$ be the 2-form

$$(37) \quad \sigma = -\frac{1}{2}\langle \text{pr}_1^* \theta^L, \text{pr}_2^* \theta^R \rangle.$$

Then the pull-back of η under group multiplication satisfies

$$\text{Mult}_H^* \eta - \text{pr}_1^* \eta - \text{pr}_2^* \eta = d\sigma.$$

Consequently, the pair (Mult_H, σ) defines a Courant morphism

$$R_{\text{Mult}_H, \sigma}: \mathbb{T}H_\eta \times \mathbb{T}H_\eta \dashrightarrow \mathbb{T}H_\eta.$$

Proposition 7.6. *The isomorphism $\mathbb{A} \rightarrow \mathbb{T}H_\eta$ intertwines the Courant morphisms*

$$\text{gr}(\text{Mult}_\mathbb{A}): \mathbb{A} \times \mathbb{A} \dashrightarrow \mathbb{A}, \quad R_{\text{Mult}_H, \sigma}: \mathbb{T}H_\eta \times \mathbb{T}H_\eta \dashrightarrow \mathbb{T}H_\eta.$$

The \mathcal{VB} -groupoid structure on $\mathbb{T}H_\eta$, defined by this isomorphism, has source and target maps

$$s(v + \alpha) = (TR_h)^*(\alpha + \frac{1}{2}\langle v, \theta_h^R \rangle), \quad t(v + \alpha) = (TL_h)^*(\alpha - \frac{1}{2}\langle v, \theta_h^L \rangle),$$

for $v + \alpha \in T_h H_\eta$. If $v_i + \alpha_i \in \mathbb{T}_{h_i} H_\eta$, $i = 1, 2$ are composable, then their groupoid product $v + \alpha = (v_1 + \alpha_1) \circ (v_2 + \alpha_2) \in \mathbb{T}_{h_1 h_2} H_\eta$ is given by $v = v_1 \circ v_2$ (product in the group TH) and

$$\alpha = (TL_{h_1^{-1}})^*(\alpha_2 - \frac{1}{2}\langle \iota(v_1)\theta_{h_1}^L, \theta_{h_2}^R \rangle).$$

Proof. The graph of the multiplication morphism for $\mathbb{T}H_\eta$ is obtained from that of $\mathbb{T}H$ as

$$R_{\text{Mult}_{H,\sigma}} = (\text{id} + \tilde{\sigma}^\#)R_{\text{Mult}_H},$$

where $\tilde{\sigma} \in \Omega^2(G \times G \times G)$ is the pull-back of σ under the map $(a, a_1, a_2) \mapsto (a_1, a_2)$. Let $H \times H$ act on $H \times H \times H$ by $(h_1, h_2)(a, a_1, a_2) = (h_1 a h_2^{-1}, h_1 a_1, a_2 h_2^{-1})$. Since $\tilde{\sigma}$ is invariant under this action, and R_{Mult_H} is invariant under its lift to the Courant algebroid, $R_{\text{Mult}_{H,\sigma}}$ is again $H \times H$ -equivariant. On the other hand, by Section 7.2 the graph $\text{gr}(\text{Mult}_{\mathbb{A}})$ of the multiplication morphism is invariant under the $H \times H$ -action on $\mathbb{A} \times \overline{\mathbb{A}} \times \overline{\mathbb{A}}$, given by

$$(h_1, h_2).(x, x_1, x_2) = ((h_1, h_2).x, (h_1, e).x_1, (e, h_2).x_2).$$

It hence suffices to check that the isomorphism $\mathbb{A}_e \rightarrow \mathbb{T}_e H_\eta$ takes $\text{gr}(\text{Mult}_{\mathbb{A}})_{(e,e,e)}$ to $R_{\text{Mult}_{H,\sigma}}|_{(e,e,e)}$. For $i = 1, 2$ let $\zeta_i \in \mathfrak{d}$ the elements corresponding to $v_i + \alpha_i \in \mathbb{T}_e H$ under the isomorphism $\mathbb{A}_e \cong \mathbb{T}_e H$, as in (36). The elements ζ_1, ζ_2 are composable if and only if $p^*\zeta_1 = s(\zeta_1) = t(\zeta_2) = (1 - p)\zeta_2$, that is (cf. (36)),

$$\frac{1}{2}p^*(v_1) + \alpha_1 = -\frac{1}{2}p^*(v_2) + \alpha_2.$$

In this case, the composition $\zeta = \zeta_1 \circ \zeta_2 = \zeta_2 + (1 - p^*)\zeta_1$ corresponds, under (36), to the element $v + \alpha$ with $v = v_1 + v_2$ and $\alpha = \alpha_2 - \frac{1}{2}p^*(v_1)$. The resulting equations read

$$v = v_1 + v_2, \quad \alpha_1 = \alpha - \frac{1}{2}p^*(v_2), \quad \alpha_2 = \alpha + \frac{1}{2}p^*(v_1).$$

Since

$$\begin{aligned} \iota(v_1, v_2)\sigma_{e,e} &= -\frac{1}{2}\langle v_1, \text{pr}_2^* \theta_e^R \rangle + \frac{1}{2}\langle \text{pr}_1^* \theta_e^L, v_2 \rangle \\ &= -\frac{1}{2}\langle p^*(v_1), \text{pr}_2^* \theta_e^R \rangle + \frac{1}{2}\langle \text{pr}_1^* \theta_e^L, p^*(v_2) \rangle \\ &= (\frac{1}{2}p^*(v_2), -\frac{1}{2}p^*(v_1)), \end{aligned}$$

these are exactly the conditions for $(v + \alpha, v_1 + \alpha_1, v_2 + \alpha_2) \in R_{\text{Mult}_{H,\sigma}}$. Consider the resulting groupoid structure on $\mathbb{T}H_\eta$. For $h = e$ and $v + \alpha \in \mathbb{T}_e H = \mathfrak{h} \oplus \mathfrak{h}^*$ we had found

$$s_e(v + \alpha) = \alpha + \frac{1}{2}p^*(v) = \alpha + \frac{1}{2}\langle v, \theta_e^R \rangle.$$

The formula at general group elements follows by right translation, using the equivariance property of the source map. The argument for the target map is similar, and the formula for groupoid multiplication is just spelling out the definition of $R_{\text{Mult}_{H,\sigma}}$. \square

APPENDIX A. COMPOSITION OF RELATIONS

For more details on the theory summarized in this section, with particular emphasis on the symplectic setting, see Guillemin-Sternberg [12].

A (linear) *relation* $R: V_1 \dashrightarrow V_2$ between vector spaces V_1, V_2 is a subspace $R \subseteq V_2 \times V_1$. Write $v_1 \sim_R v_2$ if $(v_2, v_1) \in R$. Any linear map $A: V_1 \rightarrow V_2$ defines a relation $\text{gr}(A)$. In particular, the identity map of V defines the diagonal relation $\text{gr}(\text{id}_V) = V_\Delta \subseteq V \times V$.

The *transpose relation* $R^\top: V_2 \rightarrow V_1$ consists of all (v_1, v_2) such that $(v_2, v_1) \in R$. We define

$$\ker(R) = \{v_1 \in V_1 \mid v_1 \sim 0\}, \quad \text{ran}(R) = \{v_2 \in V_2 \mid \exists v_1 \in V_1: (v_2, v_1) \in R\}$$

Given another relation $R': V_2 \dashrightarrow V_3$, the composition $R' \circ R: V_1 \dashrightarrow V_3$ consists of all (v_3, v_1) such that $v_1 \sim_R v_2$ and $v_2 \sim_{R'} v_3$ for some $v_2 \in V_2$.

We let $\text{ann}^\natural(R): V_1^* \rightarrow V_2^*$ be the relation such that $\mu_1 \sim_{\text{ann}^\natural(R)} \mu_2$ if $\langle \mu_1, v_1 \rangle = \langle \mu_2, v_2 \rangle$ whenever $v_1 \sim_R v_2$. Thus $(\mu_2, \mu_1) \in \text{ann}^\natural(R) \Leftrightarrow (\mu_2, -\mu_1) \in \text{ann}(R)$. Note $\text{ann}^\natural(V_\Delta) = (V^*)_\Delta$, and more generally

$$(38) \quad \text{ann}^\natural(\text{gr}(A)) = \text{gr}(A^*)^\top$$

for linear maps $A: V_1 \rightarrow V_2$. Suppose W_1, W_2 are vector spaces with non-degenerate symmetric bilinear forms. A relation $L: W_1 \dashrightarrow W_2$ is called *Lagrangian* if $L \subseteq W_2 \times \overline{W_1}$ is a Lagrangian subspace, where $\overline{W_1}$ indicates W_1 with the opposite bilinear form.

Lemma A.1. *If $L: W_1 \dashrightarrow W_2$ and $L': W_2 \dashrightarrow W_3$ are Lagrangian relations, then $L' \circ L: W_1 \dashrightarrow W_3$ is a Lagrangian relation.*

The analogous result for symplectic vector spaces is proved in detail in [12]; this proof carries over to Lagrangian spaces for vector spaces with split bilinear form.

Lemma A.2. *For any relations $R: V_1 \rightarrow V_2$ and $R': V_2 \rightarrow V_3$, one has $\text{ann}^\natural(R' \circ R) = \text{ann}^\natural(R') \circ \text{ann}^\natural(R)$.*

Proof. Let $W_i = V_i \oplus V_i^*$ with the metric given by the pairing $\langle (v, \alpha), (v', \alpha') \rangle = \langle \alpha, v' \rangle + \langle \alpha', v \rangle$. By Lemma A.1, the composition of Lagrangian relations

$$(R' \oplus \text{ann}^\natural(R')) \circ (R \oplus \text{ann}^\natural(R)) = (R' \circ R) \oplus (\text{ann}^\natural(R') \circ \text{ann}^\natural(R)).$$

is again a Lagrangian relation. This means that ann^\natural of the first summand is equal to the second summand. \square

The composition $R' \circ R$ can be regarded as the image of

$$R' \diamond R := (R' \times R) \cap (V_3 \times (V_2)_\Delta \times V_1)$$

under the projection to $V_3 \times V_1$.

Lemma A.3.

$$\dim(R' \diamond R) = \dim R' + \dim R - \dim V_2 + \dim(\ker(\text{ann}^\natural(R')) \cap \ker(\text{ann}^\natural(R)^\top)),$$

$$\dim(R' \circ R) = \dim(R' \diamond R) - \dim(\ker(R') \cap \ker(R^\top)).$$

Proof. The codimension of $(R' \times R) + (V_3 \times (V_2)_\Delta \times V_1)$ equals the dimension of its annihilator. It is thus equal to the dimension of

$$(\text{ann}^\natural(R') \times \text{ann}^\natural(R)) \cap (0 \times (V_2^*)_\Delta \times 0) \cong \ker(\text{ann}^\natural(R')) \cap \ker(\text{ann}^\natural(R)^\top).$$

This gives the formula for $\dim(R' \diamond R)$. On the other hand, the projection $R' \diamond R \rightarrow R' \circ R$ has kernel the intersection $(R' \times R) \cap (0 \times (V_2)_\Delta \times 0) \cong \ker(R') \cap \ker(R^\top)$. \square

The composition of linear relations R, R' is called *transverse* if

$$\ker(R') \cap \ker(R^\top) = 0, \quad \ker(\text{ann}^\natural(R')) \cap \ker(\text{ann}^\natural(R)^\top) = 0.$$

The first condition is equivalent to the claim that for $(v_3, v_1) \in R' \circ R$, there is a unique $v_2 \in V_2$ such that $(v_3, v_2) \in R'$ and $(v_2, v_1) \in R$. The second condition is equivalent to the transversality of $R' \times R$ with $V_3 \times (V_2)_\Delta \times V_1$. For transverse compositions, $R' \circ R$ varies smoothly with R', R . Either of the two conditions in the transversality condition can be replaced with the dimension formula $\dim(R' \circ R) = \dim(R') + \dim(R) - \dim V_2$. For Lagrangian relations, the dimension formula is automatic.

More generally, consider (non-linear) relations between manifolds. Here, ‘clean composition’ hypotheses are needed. Recall that the intersection of submanifolds $S_1, S_2 \subseteq M$ is *clean* (in the sense of Bott) if $S_1 \cap S_2$ is a submanifold, and $T(S_1 \cap S_2) = TS_1 \cap TS_2$. Equivalently, the intersection is clean if at all points $x \in S_1 \cap S_2$, there are local coordinates in which both S_1, S_2 are given as subspaces [14, page 491]. We say that the composition $R' \circ R$ of submanifolds $R \subseteq M_2 \times M_1$ and $R' \subseteq M_3 \times M_2$ is *clean* if

$$(39) \quad R' \diamond R = (R' \times R) \cap (M_3 \times (M_2)_\Delta \times M_1)$$

is a clean intersection, and the map $R' \diamond R \rightarrow M_3 \times M_1$ (forgetting the M_2 -component) has constant rank. Thus $R' \circ R$ is an (immersed) submanifold, and the map $R' \diamond R \rightarrow R' \circ R$ is a submersion.

The composition is called *transverse* if the composition of tangent spaces is transverse everywhere. In this case $R' \diamond R$ is a smooth submanifold of dimension $\dim R' + \dim R - \dim M_2$, and the map $R' \diamond R \rightarrow R' \circ R$ is a covering.

APPENDIX B. MATCHED PAIRS AND \mathcal{LA} -GROUPOIDS

A \mathcal{VB} -groupoid V over H is *vacant* if $V^{(0)} = V|_{H^{(0)}}$. Equivalently, the source map is a fiberwise isomorphism. In [22, 23], Mackenzie interpreted a vacant \mathcal{LA} -groupoid $E \rightrightarrows \mathfrak{g}$ over a group $H \rightrightarrows \text{pt}$ as a matched pair between a Lie algebra \mathfrak{g} and a Lie group H .² In this Section we review and elaborate these results, proving Proposition 5.1, in particular.

Lemma B.1. *Suppose $E \rightrightarrows \mathfrak{g}$ is an \mathcal{LA} -groupoid over $H \rightrightarrows \text{pt}$. Then the map*

$$(\mathbf{a}, t, s): E \rightarrow TH \times (\mathfrak{g} \oplus \mathfrak{g}), \quad x \mapsto (\mathbf{a}(x), t(x), s(x))$$

is a homomorphism of \mathcal{LA} -groupoids. If E is vacant, then (\mathbf{a}, t, s) is an embedding as a subbundle.

Proof. Since $\mathbf{a}: E \rightarrow TH$, $s: E \rightarrow \mathfrak{g}$, $t: E \rightarrow \mathfrak{g}$ are morphisms of Lie algebroids, the map (\mathbf{a}, t, s) is one also. Furthermore, if $s(x_1) = t(x_2)$,

$$\begin{aligned} (\mathbf{a}(x_1 \circ x_2), t(x_1 \circ x_2), s(x_1 \circ x_2)) &= (\mathbf{a}(x_1) \circ \mathbf{a}(x_2), t(x_1), s(x_1)) \\ &= (\mathbf{a}(x_1), t(x_1), s(x_1)) \circ (\mathbf{a}(x_2), t(x_2), s(x_2)) \end{aligned}$$

shows that (\mathbf{a}, t, s) is a \mathcal{VB} -groupoid homomorphism. If E is vacant, so that t is a fiberwise isomorphism, the map $E \rightarrow H \times \mathfrak{g}$ taking $x \in E_h$ to $(h, t(x))$ is an isomorphism. In particular, (\mathbf{a}, t, s) is an embedding as a subbundle. \square

²Note that in [22], Mackenzie uses the terminology of an *interaction* rather than a matched pair.

Proposition B.2. *Let $E \rightrightarrows \mathfrak{g}$ be a vacant \mathcal{LA} -groupoid. For any $\xi \in \mathfrak{g}$ and $h \in H$, let $h^{-1} \bullet \xi \in \mathfrak{g}$ and $\varrho(\xi)_h \in T_h H$ be defined by the condition that there exist $x \in E_h$ with*

$$(40) \quad (\mathbf{a}(x), t(x), s(x)) = (\varrho(\xi)_h, h \bullet \xi, \xi).$$

Then the map $(h, \xi) \mapsto h \bullet \xi$ defines an action of H on \mathfrak{g} , while $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(H)$ is an action of \mathfrak{g} on H . These actions satisfy the compatibility conditions,

$$(41) \quad h \bullet [\xi_1, \xi_2] = [h \bullet \xi_1, h \bullet \xi_2] + \mathcal{L}_{\varrho(\xi_1)}(h \bullet \xi_2) - \mathcal{L}_{\varrho(\xi_2)}(h \bullet \xi_1)$$

and

$$(42) \quad \varrho(\xi)_{h_1 h_2} = (\text{Mult}_H)_*(\varrho(h_2 \bullet \xi)_{h_1}, \varrho(\xi)_{h_2}).$$

Conversely, given a pair of actions of H on \mathfrak{g} and of \mathfrak{g} on H , satisfying (41) and (42), the span of the sections $\alpha(\xi)$ is a vacant \mathcal{LA} -subgroupoid.

Proof. For $h \in H$, $\xi \in \mathfrak{g}$ let $\alpha(\xi)_h = (\varrho(\xi)_h, h \bullet \xi, \xi)$ be the right hand side of (40). Since the image of E under (\mathbf{a}, t, s) is a subgroupoid, we have

$$\alpha(h_2 \bullet \xi)_{h_1} \circ \alpha(\xi)_{h_2} = \alpha(\xi)_{h_1 h_2}.$$

Applying \mathbf{a}, s to this identity gives (42) and the action property $(h_1 h_2) \bullet \xi = h_1 \bullet h_2 \bullet \xi$. On the other hand, since E is a Lie subalgebroid, $[\alpha(\xi_1), \alpha(\xi_2)] = \alpha([\xi_1, \xi_2])$.

Application of s, \mathbf{a} gives (41) and the property $[\varrho(\xi_1), \varrho(\xi_2)] = \varrho([\xi_1, \xi_2])$. Conversely, given actions ϱ and \bullet satisfying (41) and (42), let E be the subbundle of $TH \times (\mathfrak{g} \oplus \mathfrak{g})$ spanned by the sections $\alpha(\xi)$. The compatibility conditions guarantee that it is a \mathcal{VB} -subgroupoid and also a Lie subalgebroid. \square

We note that (41) and (42) are exactly the compatibility conditions for a *matched pair* $\mathfrak{g} \bowtie H$ between a Lie group and a Lie algebra as given in [22]. Therefore Proposition B.2 can be interpreted as proving a 1-1 correspondence between such matched pairs and vacant \mathcal{LA} -groupoids over a Lie group.

By differentiating the action of H on \mathfrak{g} , we obtain a linear representation of \mathfrak{h} on \mathfrak{g} (still denoted \bullet). Similarly, since $\varrho(\xi)_e = 0$, we may linearize the action of \mathfrak{g} on H to obtain a linear representation $\dot{\varrho}$ of \mathfrak{g} on \mathfrak{h} . Concretely,

$$\dot{\varrho}(\xi)(\tau) = [\varrho(\xi), \tilde{\tau}]|_e$$

where $\tilde{\tau} \in \mathfrak{X}(H)$ with $\tilde{\tau}|_e = \tau$. By linearizing (41) and (42), one obtains the following conditions, for all $\xi, \xi_1, \xi_2 \in \mathfrak{g}$ and $\tau, \tau_1, \tau_2 \in \mathfrak{h}$:

$$(43) \quad \tau \bullet [\xi_1, \xi_2] = [\tau \bullet \xi_1, \xi_2] - [\tau \bullet \xi_2, \xi_1] - \dot{\varrho}(\xi_1)(\tau) \bullet \xi_2 + \dot{\varrho}(\xi_2)(\tau) \bullet \xi_1,$$

$$(44) \quad \dot{\varrho}(\xi)([\tau_1, \tau_2]) = [\dot{\varrho}(\xi)(\tau_1), \tau_2] - [\dot{\varrho}(\xi)(\tau_2), \tau_1] - \varrho(\tau_1 \bullet \xi)(\tau_2) + \varrho(\tau_2 \bullet \xi)(\tau_1).$$

These are exactly the compatibility conditions for a *matched pair* of Lie algebras, as studied in [26, 18]. The conditions are equivalent to the statement that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ carries a Lie bracket, with $\mathfrak{g}, \mathfrak{h}$ as Lie subalgebras and such that

$$(45) \quad [\xi, \tau] = \dot{\varrho}_\xi(\tau) - \tau \bullet \xi, \quad \xi \in \mathfrak{g}, \tau \in \mathfrak{h}.$$

Let $p = p_{\mathfrak{h}}: \mathfrak{d} \rightarrow \mathfrak{h}$ be the projection to the second summand, and put $q = 1 - p$.

Proposition B.3. (a) The adjoint action $\text{Ad}: H \rightarrow \text{End}(\mathfrak{h})$ admits a unique extension $\text{Ad}: H \rightarrow \text{End}(\mathfrak{d})$ with the property

$$(46) \quad \text{Ad}_h \xi = h \bullet \xi + \iota(\varrho(\xi))\theta_h^R$$

for all $h \in H$, $\xi \in \mathfrak{g}$. Its derivative is the adjoint action of \mathfrak{h} on \mathfrak{d} .

(b) The action of \mathfrak{g} on H combines with the \mathfrak{h} -action $\nu \mapsto \nu^L$ to an action of the Lie algebra \mathfrak{d} . We have

$$\iota(\varrho(\zeta))\theta_h^R = p(\text{Ad}_h \zeta), \quad \zeta \in \mathfrak{d}.$$

(c) The action Ad_h on \mathfrak{d} is a Lie algebra automorphism of \mathfrak{d} .

Proof. (a) For $h_1, h_2 \in H$,

$$\begin{aligned} \text{Ad}_{h_1} \text{Ad}_{h_2} \xi &= \text{Ad}_{h_1} (h_2 \bullet \xi + \iota(\varrho(\xi))\theta_{h_2}^R) \\ &= h_1 h_2 \bullet \xi + \iota(\varrho(h_2 \bullet \xi))\theta_{h_1}^R + \iota(\varrho(\xi)) \text{Ad}_{h_1} \theta_{h_2}^R \\ &= h_1 h_2 \bullet \xi + \iota(\varrho(h_2 \bullet \xi), \varrho(\xi))(\text{Mult}^* \theta^R)_{h_1, h_2} \\ &= h_1 h_2 \bullet \xi + \iota(\varrho(\xi))\theta_{h_1 h_2}^R \end{aligned}$$

where we used (41). This shows that (46) extends the adjoint action on \mathfrak{h} to an action on \mathfrak{d} . It is clear that Ad_h extends the adjoint action on \mathfrak{h} , and that $\frac{\partial}{\partial t} \text{Ad}_{t\tau} \xi = [\tau, \xi]$ for $\xi \in \mathfrak{g}$, $\tau \in \mathfrak{h}$ (cf. (45)).

(b) Note first that for $\tau \in \mathfrak{h}$, $\varrho(\tau) = \tau^L$. On the other hand, for $\xi \in \mathfrak{h}$ the formula $\iota(\varrho(\xi))\theta_h^R = p(\text{Ad}_h \xi)$ follows from (46), by applying p to both sides. For $\tau, \xi \in \mathfrak{h}$, $\xi \in \mathfrak{g}$ we compute:

$$\iota([\varrho(\tau), \varrho(\xi)])\theta^R = \mathcal{L}(\tau^L)\iota(\varrho(\xi))\theta^R = \mathcal{L}(\tau^L)(p(\text{Ad}_h \xi)) = p(\text{Ad}_h [\tau, \xi]) = \iota(\varrho([\tau, \xi]))\theta^R$$

hence $[\varrho(\tau), \varrho(\xi)] = \varrho([\tau, \xi])$.

(c) We have $\text{Ad}_h [\tau, \xi] = [\text{Ad}_h \tau, \text{Ad}_h \xi]$, $\xi \in \mathfrak{g}$, $\tau \in \mathfrak{h}$, by taking the derivative of $\text{Ad}_h \text{Ad}_{\exp(t\tau)} \xi = \text{Ad}_{\exp(t \text{Ad}_h \tau)} \text{Ad}_h \xi$. For $\xi_1, \xi_2 \in \mathfrak{g}$, we have

$$\mathcal{L}_{\varrho(\xi_1)}(\text{Ad}_h \xi_2) = [\iota(\varrho(\xi_1))\theta^R, \text{Ad}_h \xi_2] = [p(\text{Ad}_h \xi_1), \text{Ad}_h \xi_2].$$

The \mathfrak{g} -component of the desired equation $\text{Ad}_h([\xi_1, \xi_2]) = [\text{Ad}_h \xi_1, \text{Ad}_h \xi_2]$ now follows from Equation (41):

$$\begin{aligned} q(\text{Ad}_h[\xi_1, \xi_2]) &= [q(\text{Ad}_h \xi_1), q(\text{Ad}_h \xi_2)] + \mathcal{L}_{\varrho(\xi_1)} q(\text{Ad}_h \xi_2) - \mathcal{L}_{\varrho(\xi_2)} q(\text{Ad}_h \xi_1) \\ &= q([q(\text{Ad}_h \xi_1), q(\text{Ad}_h \xi_2)] + [p(\text{Ad}_h \xi_1), \text{Ad}_h \xi_2] - [p(\text{Ad}_h \xi_2), \text{Ad}_h \xi_1]) \\ &= q([\text{Ad}_h \xi_1, \text{Ad}_h \xi_2]). \end{aligned}$$

Similarly, the \mathfrak{h} -component is obtained by the calculation,

$$\begin{aligned} p(\text{Ad}_h[\xi_1, \xi_2]) &= \iota(\varrho([\xi_1, \xi_2]))\theta^R \\ &= \mathcal{L}_{\varrho(\xi_1)}\iota(\varrho(\xi_2))\theta^R - \mathcal{L}_{\varrho(\xi_2)}\iota(\varrho(\xi_1))\theta^R - [\iota(\varrho(\xi_1))\theta^R, \iota(\varrho(\xi_2))\theta^R] \\ &= \mathcal{L}_{\varrho(\xi_1)}p(\text{Ad}_h \xi_2) - \mathcal{L}_{\varrho(\xi_2)}p(\text{Ad}_h \xi_1) - [p(\text{Ad}_h \xi_1), p(\text{Ad}_h \xi_2)] \\ &= p([p(\text{Ad}_h \xi_1), \text{Ad}_h \xi_2]) - p([p(\text{Ad}_h \xi_2), \text{Ad}_h \xi_1]) - [p(\text{Ad}_h \xi_1), p(\text{Ad}_h \xi_2)] \\ &= p([\text{Ad}_h \xi_1, \text{Ad}_h \xi_2]). \end{aligned}$$

□

Proposition B.3 shows that a vacant \mathcal{LA} -groupoid over $H \rightrightarrows \text{pt}$ determines an H -equivariant Lie algebra triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$, as in Definition 3.2. The converse was established in Proposition 3.3.

Using the H -action on \mathfrak{d} we can now characterize E directly in terms of the H -equivariant triple.

Proposition B.4. *Suppose $E \rightarrow H$ is a vacant \mathcal{LA} -groupoid over a group H , and define the H -equivariant triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ as explained above. Then*

$$(a, t, s)(E) = \{(v, \xi, \xi') \mid v \in T_h H, \xi, \xi' \in \mathfrak{g}, \text{Ad}_h \xi' - \xi = \iota(v)\theta_h^R\}$$

Proof. The condition $\text{Ad}_h \xi' - \xi = \iota(v)\theta_h^R$ is just (46), proving the inclusion \subseteq . The opposite inclusion follows by dimension count: Given $\xi' \in \mathfrak{g}$, the elements ξ, v are determined as $\xi = q(\text{Ad}_h \xi')$, $\iota(v)(\theta_h^R) = p(\text{Ad}_h \xi)$. □

The correspondence between \mathcal{LA} -groupoids and H -equivariant triples is compatible with morphisms. Suppose $E_i \rightrightarrows \mathfrak{g}_i$, $i = 0, 1$ are vacant \mathcal{LA} -groupoids over groups H_i . A *morphism* (resp. *comorphism*) from E_0 to E_1 is a Lie group homomorphism $\Phi: H_0 \rightarrow H_1$, together with a vector bundle map $E_0 \rightarrow E_1$ (resp. $\Phi^* E_1 \rightarrow E_0$) whose graph is an \mathcal{LA} -subgroupoid of $E_1 \times E_0$ along the graph of Φ . If E_i are vacant, so that $E_i|_e \cong \mathfrak{g}_i$, such a morphism (resp. comorphism) defines a pair of Lie algebra homomorphisms $d_e \Phi: \mathfrak{h}_0 \rightarrow \mathfrak{h}_1$ and $\mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ (resp. $\mathfrak{g}_1 \rightarrow \mathfrak{g}_0$). Let $\mathfrak{d}_i = \mathfrak{g}_i \oplus \mathfrak{h}_i$.

Proposition B.5. (a) *If $E_0 \rightarrow E_1$ is a morphism of \mathcal{LA} -groupoids, then the linear map $\mathfrak{d}_0 \rightarrow \mathfrak{d}_1$ given as the direct sum of the maps $\mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ and $\mathfrak{h}_0 \rightarrow \mathfrak{h}_1$ is a Lie algebra homomorphism, equivariant relative to the underlying group homomorphism $\Phi: H_0 \rightarrow H_1$.*
 (b) *If $\Phi^* E_1 \rightarrow E_0$ is a comorphism of \mathcal{LA} -groupoids, then the subspace $\mathfrak{r} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$, given as the direct sum of the graphs of the maps $\mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ and $\mathfrak{h}_0 \rightarrow \mathfrak{h}_1$, is a Lie subalgebra, invariant under the action of H_0 (via its inclusion as $\text{gr}(\Phi) \subseteq H_1 \times H_0$).*

Proof. (a) The statement is obvious if $E_0 \rightarrow E_1$ is an inclusion. The general case reduces to that of an inclusion, by letting $E'_1 = E_1 \times E_0$, $H'_1 = H_1 \times H_0$, $\Phi'(h_0) = (\Phi(h_0), h_0)$, and with the inclusion $E_0 \rightarrow E'_1$ the direct sum of the identity map with the map $E_0 \rightarrow E_1$.

(b) Similar to (a), let $E'_0 \hookrightarrow E'_1 = E_1 \times E_0$ be the inclusion of the graph of the map $\Phi^* E_1 \rightarrow E_0$. By (a), applied to inclusions one obtains an $H_0 \cong \text{gr}(\Phi)$ -equivariant Lie algebra homomorphism $\mathfrak{d}'_0 := \mathfrak{g}_1 \times \mathfrak{h}_0 \hookrightarrow \mathfrak{d}'_1 := \mathfrak{d}_1 \times \mathfrak{d}_0$. Its range is \mathfrak{r} , which is hence an H_0 -invariant Lie subalgebra. □

Conversely, given an $H_0 \cong \text{gr}(\Phi)$ -invariant Lie subalgebra $\mathfrak{r} \subseteq \mathfrak{d}_1 \times \mathfrak{d}_0$ given as the direct sum of the graphs of $d_e \Phi$ and the graph of a Lie algebra homomorphism $\mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ (resp. $\mathfrak{g}_1 \rightarrow \mathfrak{g}_0$), the resulting \mathcal{LA} -subgroupoid of $E_1 \times E_0$ defines a morphism (resp. comorphism) of \mathcal{LA} -groupoids.

APPENDIX C. SOME CONSTRUCTIONS WITH \mathcal{VB} -GROUPOIDS

We will base our discussion on the following result

Proposition C.1. [29, § 5.3] *Suppose H, G, K are Lie groupoids, and $\phi: G \rightarrow K$ and $\psi: H \rightarrow K$ are morphisms of Lie groupoids. If ϕ and ψ are transverse, then the fibered product $H \times_K G$ is a Lie groupoid, with $H^{(0)} \times_{K^{(0)}} G^{(0)}$ as its space of units.*

Remark C.2. Note that, [29, § 5.3] makes the additional assumption that the restrictions $\phi^{(0)} = \phi|_{G^{(0)}}$ and $\psi^{(0)} = \psi|_{H^{(0)}}$ to the units are transverse. However, this property is automatic: Suppose $x \in G^{(0)}$, $y \in H^{(0)}$ are units with $w := \phi(x) = \psi(y)$. Then

$$T_x G = \ker(T_x s) \oplus T_x G^0, \quad T_y H = \ker(T_y s) \oplus T_y H^0, \quad T_w K = \ker(T_w s) \oplus T_w K^0.$$

Since the tangent maps to ϕ , ψ respect these decompositions, their transversality implies that of $\phi^{(0)}$, $\psi^{(0)}$.

Corollary C.3. *Suppose $\phi: V \rightarrow W$ is a fiberwise surjective homomorphism of \mathcal{VB} -groupoids over $G \rightarrow H$. Then $\ker(\phi)$ is a \mathcal{VB} -subgroupoid of V .*

Proof. Since ϕ is fiberwise surjective, it is transverse to the zero section $H \rightarrow W$. We may view $\ker(\phi)$ as the fibered product $V \times_W H$, where $H \rightarrow W$ is the inclusion of the zero section. By Proposition C.1, it is a Lie groupoid. By Definition 2.1 it is a \mathcal{VB} -groupoid. \square

For the next result, we recall Pradines' observation [33] (see also [24, § 11.2]) that the dual of a \mathcal{VB} -groupoid $V \rightarrow H$ carries a natural structure of \mathcal{VB} -groupoid,

$$\begin{array}{ccc} V^* & \rightrightarrows & \text{ann}(V^{(0)}) \\ \downarrow & & \downarrow \\ H & \rightrightarrows & H^{(0)} \end{array}$$

where $\text{ann}(V^{(0)})$ is the annihilator of $V^{(0)}$ in $V^*|_{H^{(0)}}$. The groupoid structure is given by $\langle \alpha_1 \circ \alpha_2, v_1 \circ v_2 \rangle = \langle \alpha_1, v_1 \rangle + \langle \alpha_2, v_2 \rangle$, for composable elements $\alpha_1, \alpha_2 \in V^*$ and $v_1, v_2 \in V^*$, with α_i having the same base points as v_i .

Alternatively, one can define the groupoid multiplication in terms of its graph by

$$(47) \quad \text{gr}(\text{Mult}_{V^*}) = \text{ann}^\natural(\text{gr}(\text{Mult}_V))$$

(using the notation from Appendix A). Writing the groupoid axioms in terms of compositions of relations, it then follows from the vector bundle version of Lemma A.2, that the \mathcal{VB} -groupoid axioms of V imply those for V^* .

Suppose now that $\Phi: V \rightarrow W$ is a morphism of \mathcal{VB} -groupoids, i.e.

$$\text{gr}(\Phi) \circ \text{Mult}_V \subseteq \text{Mult}_W \circ \text{gr}(\Phi \times \Phi).$$

By application of Lemma A.2 and (47) one obtains the corresponding equation for $\Phi^*: W^* \rightarrow V^*$ holds. Thus we have proven [24, Proposition 11.2.6], that the dual bundle map $\Phi^*: W^* \rightarrow V^*$ is again a morphism of \mathcal{VB} -groupoids.

Corollary C.4. *Suppose $C \subseteq V$ is a \mathcal{VB} -subgroupoid over groupoids $K \subseteq H$. Then $\text{ann}(C) \subseteq V^*$ is a \mathcal{VB} -subgroupoid. Its space of objects is $\text{ann}(C) \cap \text{ann}(V^{(0)})$.*

Proof. Let $i: K \hookrightarrow H$ be inclusion. By Proposition C.1, the pull-back $i^*V^* \rightarrow K$ is a \mathcal{VB} -groupoid. It comes with a fiberwise surjective Lie groupoid homomorphism $i^*V^* \rightarrow C^*$, where the map on units is again fiberwise surjective. Its kernel is $\text{ann}(C)$. \square

A non-degenerate fiber metric $\langle \cdot, \cdot \rangle$ on a \mathcal{VB} -groupoid V is *multiplicative* if it satisfies

$$(48) \quad \langle v_1 \circ v_2, v'_1 \circ v'_2 \rangle = \langle v_1, v'_1 \rangle + \langle v_2, v'_2 \rangle$$

for composable elements v_1, v_2 , resp. v'_1, v'_2 , with v_i having the same base points as v'_i . Equivalently, the graph $\text{gr}(\text{Mult}_V) \subset V \times \overline{V} \times \overline{V}$ is an isotropic subbundle, where \overline{V} denotes V with the opposite fiber metric.

The fiber metric $\langle \cdot, \cdot \rangle$ defines a map $\Psi: V \rightarrow V^*$, and (48) shows that

$$(49) \quad \Psi(\text{gr}(\text{Mult}_V)) \subseteq \text{gr}(\text{Mult}_{V^*}) = \text{ann}^\natural(\text{gr}(\text{Mult}_V)).$$

Since, in addition, the fiber metric $\langle \cdot, \cdot \rangle$ is non-degenerate, Ψ defines an isomorphism of \mathcal{VB} -groupoids. This shows that (49) is an equality and

$$\Psi(V^{(0)}) = (V^*)^{(0)} = \text{ann}(V^{(0)}).$$

Therefore both $\text{gr}(\text{Mult}_V)$ and $V^{(0)}$ are Lagrangian.

Corollary C.5. *Let $V \rightarrow H$ be a \mathcal{VB} -groupoid, equipped with a multiplicative non-degenerate fiber metric. Let $C \rightrightarrows C \cap V^{(0)}$ be a co-isotropic \mathcal{VB} -subgroupoid. Then $C^\perp \rightrightarrows C^\perp \cap V^{(0)}$ is a \mathcal{VB} -subgroupoid of C , and hence the quotient inherits a \mathcal{VB} -groupoid structure $C/C^\perp \rightrightarrows (C \cap V^{(0)})/(C^\perp \cap V^{(0)})$. Moreover, the natural non-degenerate fiber metric on C/C^\perp is multiplicative.*

Proof. The identification $V^* \cong V$ identifies $\text{ann}(C) \cong C^\perp \subseteq C$. By the previous Corollary, this is a \mathcal{VB} -subgroupoid of V . Hence $C^\perp \rightarrow C$ is an inclusion of \mathcal{VB} -groupoids. Therefore, the dual morphism,

$$(50) \quad C^* \rightarrow (C^\perp)^*,$$

is a surjective submersion of \mathcal{VB} -groupoids. Thus, by Corollary C.3, the kernel $(C/C^\perp)^* \cong C/C^\perp$ of (50) carries a natural \mathcal{VB} -groupoid structure. Finally, it is clear that the restriction of the fiber metric $\langle \cdot, \cdot \rangle$ to C/C^\perp satisfies (48), since it does so for V . \square

REFERENCES

- [1] Anton Alekseev, Henrique Bursztyn, and Eckhard Meinrenken, *Pure spinors on Lie groups*, Astérisque (2009), no. 327, 131–199 (2010).
- [2] Anton Alekseev and Yvette Kosmann-Schwarzbach, *Manin pairs and moment maps*, Journal of Differential Geometry **56** (2000), no. 1, 133–165.
- [3] Anton Alekseev, Anton Malkin, and Eckhard Meinrenken, *Lie group valued moment maps*, Journal of Differential Geometry, **48** (1998), no. 3, 445–495.
- [4] Anton Alekseev and Ping Xu, *Derived brackets and Courant algebroids*, Available at <http://www.math.psu.edu/ping/anton-final.pdf>, 2002.
- [5] Henrique Bursztyn, David Iglesias Ponte, and Pavol Ševera, *Courant morphisms and moment maps*, Mathematical Research Letters **16** (2009), no. 2, 215–232.
- [6] Theodore James Courant, *Dirac manifolds*, Transactions of the American Mathematical Society **319** (1990), no. 2, 631–661.
- [7] Irene Dorfman, *Dirac structures and integrability of nonlinear evolution equations*, John Wiley & Son Ltd, July 1993.

- [8] Vladimir Gershonovich Drinfel'd, *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations*, Doklady Akademii Nauk SSSR **268** (1983), no. 2, 285–287.
- [9] ———, *Quasi-Hopf Algebras*, Algebra i Analiz, 1989.
- [10] Janusz Grabowski and Mikolaj Rotkiewicz, *Higher vector bundles and multi-graded symplectic manifolds*, Journal of Geometry and Physics **59** (2009), no. 9, 1285–1305.
- [11] Marco Gualtieri, *Generalized complex geometry*, Ph.D. thesis, University of Oxford, November 2003.
- [12] Victor Guillemin and Shlomo Sternberg, *Semi-classical analysis*, Available at <http://www-math.mit.edu/~vvg/semiclassGuilleminSternberg.pdf>.
- [13] Nigel Hitchin, *Generalized Calabi-Yau manifolds*, The Quarterly Journal of Mathematics **54** (2003), no. 3, 281–308.
- [14] Lars Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer-Verlag, 1990.
- [15] Madeleine Jotz, *Dirac Lie groups, Dirac homogeneous spaces and the Theorem of Drinfeld*, (2009), Available at <http://arxiv.org/abs/0910.1538>.
- [16] Ctirad Klimčík and Pavol Ševera, *Open strings and D-branes in WZNW models*. Nuclear Phys. B 488 (1997), no. 3, 653–676.
- [17] Yvette Kosmann-Schwarzbach, *Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory*. The breadth of symplectic and Poisson geometry (2005).
- [18] Yvette Kosmann-Schwarzbach and Franco Magri, *Poisson-Lie groups and complete integrability. I. Drinfel'd bialgebras, dual extensions and their canonical representations*, Annales de l'Institut Henri Poincaré **49** (1988), no. 4, 433–460.
- [19] David Li-Bland and Eckhard Meinrenken, *Courant algebroids and Poisson geometry*, International Mathematics Research Notices **2009** (2009), no. 11, 2106–2145.
- [20] David Li-Bland and Pavol Ševera, *Quasi-Hamiltonian groupoids and multiplicative Manin pairs*, International Mathematics Research Notices **2011** (2011), no. 10, 2295–2350.
- [21] Zhang-Ju Liu, Alan Weinstein, and Ping Xu, *Manin triples for Lie bialgebroids*, Journal of Differential Geometry **45** (1997), no. 3, 547–574.
- [22] Kirill C. H. Mackenzie, *Double Lie algebroids and second-order geometry. I*, Advances in Mathematics **94** (1992), no. 2, 180–239.
- [23] ———, *Double Lie algebroids and second-order geometry. II*, Advances in Mathematics **154** (2000), no. 1, 46–75.
- [24] ———, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005.
- [25] Kirill C. H. Mackenzie and Ping Xu, *Lie bialgebroids and Poisson groupoids*, Duke Mathematical Journal **73** (1994), no. 2, 415–452.
- [26] Shahn Majid, *Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations*, Pacific Journal of Mathematics **141** (1990), no. 2, 311–332.
- [27] Rajan Amit Mehta, *Q-groupoids and their cohomology*, Pacific Journal of Mathematics **242** (2009), no. 2, 311–332.
- [28] Brett Milburn, *Two Categories of Dirac Manifolds*. Available at <http://arxiv.org/abs/0712.2636>.
- [29] Ieke Moerdijk and Janez Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge University Press, 2003.
- [30] Cristián Ortiz, *Multiplicative Dirac structures on Lie groups*, Comptes Rendus Mathématique **346** (2008), no. 23-24, 1279–1282.
- [31] ———, *Multiplicative Dirac structures*, Ph.D. thesis, Instituto de Matemática Pura e Aplicada, April 2009.
- [32] David Iglesias Ponte and Ping Xu, *Hamiltonian spaces for Manin pairs over manifolds*, (2008), Available at <http://arxiv.org/abs/0809.4070>.
- [33] Jean Pradines, *Remarque sur le groupoïde cotangent de Weinstein-Dazord*, Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique **306** (1988), no. 13, 557–560.
- [34] Dmitry Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, Ph.D. thesis, University of California, Berkeley, 1999.

- [35] ———, *Quasi-Lie bialgebroids and twisted Poisson manifolds*, Letters in Mathematical Physics **61** (2002), no. 2, 123–137.
- [36] Pavol Ševera, *Letters to A. Weinstein*, Available at <http://sophia.dtp.fmph.uniba.sk/~severa/letters/>.
- [37] Michal Siran, *On non-connected Poisson-Lie groups*.
- [38] Kyouzuke Uchino, *Remarks on the definition of a Courant algebroid*, Letters in Mathematical Physics. A Journal for the Rapid Dissemination of Short Contributions in the Field of Mathematical Physics **60** (2002), no. 2, 171–175.

UNIVERSITY OF TORONTO, DEPARTMENT OF MATHEMATICS, 40 ST GEORGE STREET, TORONTO, ONTARIO M4S2E4, CANADA

E-mail address: `dbland@math.toronto.edu`

UNIVERSITY OF TORONTO, DEPARTMENT OF MATHEMATICS, 40 ST GEORGE STREET, TORONTO, ONTARIO M4S2E4, CANADA

E-mail address: `mein@math.toronto.edu`